PRINCIPLES OF MECHANISM

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PREFACE TO THE FOURTH EDITION

IN many Universities the final examinations for a degree in Engineering are taken in two or more parts and Part I is generally regarded as common to all students of engineering. After Part I they tend to spend more time in the specialised departments of the University in Aeronautical, Civil, Electrical or Mechanical Engineering. Some students thus cease to study a particular subject after Part I and it is controversial whether a single textbook on any special subject can cope adequately with the requirements of a student studying that subject for both or all parts of his final examination. After careful consideration of the desirability of enlarging the third edition to cover more advanced work it has been decided to confine the contents of the fourth edition approximately to Part I. The Appendix has been revised and enlarged and a system of notation adopted for the Coriolis component of acceleration which, it is hoped, will make the different components of acceleration easier of interpretation.

F. DYSON

SOUTH KENSINGTON
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PREFACE TO THE THIRD EDITION

THE general arrangement of the book remains unchanged, but new sections dealing with more advanced work have been added. These additions include: an analysis of the sun and planet epicyclic gear; pre-selective epicyclic gear box; exact expression for the acceleration of a piston; mathematical expressions for displacement, velocity, and acceleration of a follower in contact with a cam face of straight or circular outline; acceleration diagram for a quick-return motion. Students reading the subject for the first time may omit these sections as they are intended for more advanced students.

The author wishes to express his thanks to correspondents for suggestions, many of which have been incorporated.

F. DYSON

SOUTH KENSINGTON
July, 1939

BECAUSE of the gratifying reception, both in the press and correspondence, it has not been considered necessary to make any serious changes from the first edition. In addition to minor corrections and changes, it is hoped that the sections on viscous friction and controlling force diagrams for governors will enhance the value of the book to students

F. DYSON

SOUTH KENSINGTON
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PREFACE TO THE FIRST EDITION

THE object of the present volume is to present to the student in a comprehensive manner those fundamental principles which apply to the moving parts of machines. This part of engineering is usually dealt with under the heading 'Theory of Machines', and the author has attempted to deal with the subject-matter in a manner particularly adapted to the needs of students. It is hoped that the volume will be suitable for students studying for an engineering degree, for those who desire to sit for examinations conducted by the Engineering Institutions, and for those studying under the joint scheme for National Certificates and Diplomas.

The author desires to thank the Senate of the University of London, the Council of the Institution of Civil Engineers, the Council of the Institution of Mechanical Engineers, and Messrs. Wm. Clowes & Sons Ltd., for permission to use questions from examination papers.

Thanks are also due to Mr. W. Collins, A.C.G.I., B.Sc., A.M.I.Mech.E., for help in the preparation of drawings, and for checking numerous exercises.

F. DYSON

SOUTH KENSINGTON
October, 1927

CONTENTS

CHAPTER I

PLANE MOTION

§ **1. Introduction.** A machine is usually understood to consist of moving parts, that is, parts moving relatively to each other. The motion of a rigid part of a machine is perfectly determinate when the motion of two points or particles of the particular part are known. Thus, in a connecting-rod of an ordinary steam engine, if the motion of the two extremities of the connecting-rod is known, the motion of any intermediate point on the connecting-rod is known. The determination of the motion of the two extremities of a connecting-rod is best accomplished by studying each of them in turn, that is, by treating each as a particle moving in a definite manner. Generally speaking, before attempting to discuss the more complex motions of the moving parts of machines it is necessary to investigate the motion of a particle in one plane.

A particle is usually understood to mean a very small body whose mass may be considered as concentrated at a point. The motion of a particle is always understood as being relative to some other body—usually the earth. Again, when a body is said to be at rest or to have no motion, the state of rest is relative to some other body—usually the earth. The earth itself has motion relative to the sun, but in the study of the motions of parts of machines the motion is usually relative to the earth, and to keep the problem confined to small limits it is usual to regard the earth as at rest.

§ **2. Velocity.** When a particle changes its position relative to a fixed body, the displacement is independent of the time taken for the change to take place. The distance traversed while the change of position is taking place is known as displacement. The velocity of a particle is the rate of change of displacement with respect to time. If a particle moves through equal displacements in equal time-intervals, the velocity is said to be uniform. Thus, a train travelling between two stations 15 miles apart may accomplish the

journey in half an hour and the average velocity may be said to be 30 miles per hour. Since the train has no velocity when starting and when finishing, the velocity throughout the journey cannot be constant, and the statement that the average velocity is 30 miles per hour gives no clear idea as to the velocity at intermediate points between the stations. It is thus clear that when a velocity of 30 miles per hour is stated it is not necessary that the train or particle should travel for 1 hour. Actually it is much more convenient to reduce the time-base to smaller dimensions to arrive at a truer estimate of velocity. A speed of 30 miles per hour is equivalent to $\frac{30 \times 5{,}280}{60 \times 60} = 44$ ft. per sec. $= 4{\cdot}4$ ft. per $0{\cdot}1$ sec. $= 0{\cdot}44$ ft. per $0{\cdot}01$ sec. By reducing the time-interval to small dimensions, the true rate of change of displacement can be more readily visualized.

As will be seen later, velocity implies direction in addition to magnitude, and the term speed is often applied to the magnitude of a velocity when the direction is relatively unimportant. The speed of a train may be stated as 40 miles per hour, the direction in which the train is moving not being significant for the problem under consideration.

§ 3. **Acceleration.** In practice, cases of constant velocity are somewhat rare, and when not constant the velocity is said to be variable. Variable velocity implies either an increasing or decreasing velocity. When the velocity of a particle is increasing, the particle is said to be accelerated; and when the velocity is decreasing, the particle is being decelerated or retarded. When dealing with change of velocity the term 'acceleration' is frequently used in a general sense to imply either increase or decrease of velocity.

Acceleration may be defined as the rate of change of velocity with respect to time. When the velocity of a train changes from, say, 30 to 35 miles per hour the time-interval during which the change of velocity takes place becomes important. Thus a change of velocity of 5 miles per hour in 10 sec. involves a greater time-rate change of velocity than the same change in 1 min. The unit of acceleration is usually

1 ft. per sec. per sec. An acceleration of 5 miles per hour in 3 min. is equivalent to $\frac{5\times 5,280}{60\times 60}$ ft. per sec. in 3 min. or $\frac{5\times 5,280}{60\times 60\times 3}$ ft. per sec. per min. or $\frac{5\times 5,280}{60\times 60\times 3\times 60}$ ft. per sec. per sec. or $\frac{11}{270}$ ft. per sec. per sec.

When the velocity is increasing it is usual to regard the acceleration as positive, and when decreasing as negative. Negative acceleration is thus equivalent to deceleration or retardation.

§ 4. **Equations of Linear Motion.** When a particle is moving in a straight path its motion is said to be linear. The relations that exist between displacement, time, velocity, and acceleration may be expressed by formulae which can easily be proved.

Let v_0 = initial velocity in feet per second,
v = final velocity after t seconds,
t = time-interval in seconds,
a = acceleration in feet per second per second,
s = displacement in feet.

Then,
$$v = v_0 + at, \tag{1}$$
$$s = v_0 t + \tfrac{1}{2}at^2, \tag{2}$$
$$v^2 = v_0^2 + 2as. \tag{3}$$

The above equations may be used when the unit of displacement is the centimetre, metre, or other convenient unit.

EXAMPLE 1. At a certain instant a particle has a velocity of 3 kilometres per hour; the deceleration is 5 in centimetre second units. Find the time that elapses before the velocity is 18 metres per minute.

3 kilometres per hour = $3\times 1,000$ metres per hour

$$= \frac{3\times 1,000\times 100}{60\times 60} \text{ centimetres per second} = v_0.$$

18 metres per minute = $\frac{18\times 100}{60}$ centimetres per second.

Using equation (1) and remembering that the acceleration is negative,
$$\frac{18\times 100}{60} = \frac{3\times 1,000\times 100}{60\times 60} - 5t,$$

or
$$5t = \frac{1,000}{12} - 30 = \frac{640}{12}.$$
$$\therefore\ t = 10\tfrac{2}{3} \text{ sec.}$$

EXAMPLE 2. A body is projected vertically with a speed of 100 ft. per sec. How long will it take to reach a point 80 ft. above the point of projection, and what time will elapse before it again passes that point? What is the speed of the body when passing this point? [*Inst. C. E.*]

Using equation (2) and taking the acceleration due to gravity (g) as 32·2 ft. per sec. per sec.,
$$80 = 100t - 16{\cdot}1t^2,$$
or
$$t^2 - \frac{100}{16{\cdot}1}t + \frac{80}{16{\cdot}1} = 0.$$
$$\therefore\ t = 0{\cdot}94 \text{ sec. or } 5{\cdot}27 \text{ sec.}$$

The time to reach the highest point of its flight is found by using equation (1):
$$0 = 100 - 32{\cdot}2t.$$
$$\therefore\ t = 3{\cdot}105 \text{ sec.}$$

Time taken to travel from 80 ft. above the point of projection to highest point $= 3{\cdot}105 - 0{\cdot}94 = 2{\cdot}165$ sec.

$\therefore$ Time that elapses before it passes the given point again
$$= 2 \times 2{\cdot}165 = 4{\cdot}33 \text{ sec.; or } 5{\cdot}27 - 0{\cdot}94 = 4{\cdot}33 \text{ sec.}$$

The speed when passing the given point is found by using equation (1):
$$v = 100 - 32{\cdot}2 \times 0{\cdot}94 = 69{\cdot}7 \text{ ft. per sec.}$$

This may be verified by using equation (3):
$$v^2 = 100^2 - 2 \times 32{\cdot}2 \times 80 = 10,000 - 5,150 = 4,850.$$
$\therefore\ v = 69{\cdot}7$ ft. per sec., which agrees with that found above.

EXAMPLE 3. A train running down a steady incline passes quarter-mile posts as follows: first post, 0 sec.; second post, 40 sec.; third post, 70 sec. Find the speed of the train when passing the third post.

Using equation (2):
$$1,320 = v_0 \times 40 + \tfrac{1}{2} \times a \times 40^2 \qquad \text{(i)}$$
$$2,640 = v_0 \times 70 + \tfrac{1}{2} \times a \times 70^2,$$
whence
$$9,240 = 280v_0 + 5,600a$$
$$10,560 = 280v_0 + 9,800a$$
$$1,320 = 4,200a.$$
$$\therefore\ a = 0{\cdot}314 \text{ ft. per sec. per sec.}$$

Substituting this value of a in (i):

$$1{,}320 = 40v_0 + 800 \times 0{\cdot}314.$$

$$\therefore\ v_0 = 26{\cdot}7 \text{ ft. per sec.}$$

The speed when passing the third post is found by using equation (1):

$$v = 26{\cdot}7 + 0{\cdot}314 \times 70 = 48{\cdot}7 \text{ ft. per sec.}$$

or
$$\frac{48{\cdot}7 \times 60 \times 60}{5{,}280} = 33{\cdot}2 \text{ miles per hour.}$$

§ **5. Circular Motion.** A particle moving in a circular path of radius r about a fixed centre moves through a distance equal to the circumference of the circle for each revolution. The length of a circular arc is the product of the radius of the circle and the angle, expressed in radians, subtended at the centre of the circle. Thus, in Fig. 1, let A and B be points on a circular path of radius r; a particle moving from A to B has a displacement of $r\theta$, whence $s = r\theta$.

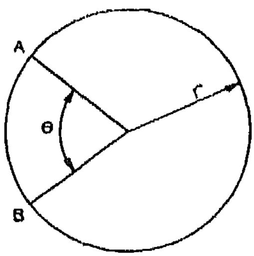


Fig. 1.

In Fig. 1, let a particle be moving in a circular path of radius r at N revolutions per minute; then the number of radians per minute $= N \times 2\pi$ and the radians per second $= \dfrac{2\pi N}{60}$. In scientific work it is usual to express angular velocities in radians per second.

Let ω = angular velocity in radians per second,

then
$$\omega = \frac{2\pi N}{60}.$$

The peripheral speed of a particle may be regarded as the distance moved through in unit time.

Let v = peripheral speed,
then $v = r\omega$ when the angular velocity is constant, since the distance moved through in unit time is $r\omega$.

Angular acceleration is usually expressed in radians per second per second, and the relation between linear or peripheral acceleration and angular acceleration is expressed by

$$a = r\alpha,$$

where α is the angular acceleration in radians per second per second.

Corresponding to the formulae for variable linear motion there are three formulae for variable circular motion which are strictly analogous.

For variable circular motion,

let ω_0 = initial angular velocity in radians per second,
ω = final angular velocity after t seconds,
t = time-interval in seconds,
α = angular acceleration in radians per second per second,
θ = angular displacement in radians,

then

$$\omega = \omega_0 + \alpha t, \tag{4}$$

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2, \tag{5}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta. \tag{6}$$

EXAMPLE 4. The wheels of a motor-car are 30 in. in diameter. Find their angular velocity in radians per second when the speed of the car is 25 miles per hour.

$$25 \text{ miles per hour} = \frac{25 \times 5{,}280}{60 \times 60} = \frac{110}{3} \text{ ft. per sec.},$$

$$\therefore \text{ angular velocity of wheels} = \frac{v}{r} = \frac{110}{3 \times 1\frac{1}{4}} = 29{\cdot}3 \text{ radians per sec.}$$

EXAMPLE 5. A fly-wheel has its speed reduced from 240 to 180 revs. per min. in $\frac{1}{2}$ min. Determine the angular retardation and the total angle turned through before coming to rest, assuming that the retardation is uniform.

Retardation is 60 revs. per min. in $\frac{1}{2}$ min., this corresponds to $\frac{60 \times 2\pi}{60 \times 30}$ or $0{\cdot}209$ radians per sec. per sec.

This may be verified by using equation (4):

$$\frac{180\times 2\pi}{60} = \frac{240\times 2\pi}{60}+\alpha\times 30,$$

or
$$\alpha = -\frac{60\times 2\pi}{60\times 30} = -0{\cdot}209 \text{ radian per sec. per sec.}$$

The negative sign indicates that the angular velocity is decreasing, i.e. the wheel is decelerating.

Using equation (6), and remembering that the final angular velocity is zero,

$$0 = \left(\frac{240\times 2\pi}{60}\right)^2 - 2\times 0{\cdot}209\times\theta,$$

whence
$$\theta = 1{,}508 \text{ radians.}$$

This may be verified as follows: the retardation is 60 revs. per min. per $\frac{1}{2}$ min., or 120 revs. per min. per min. The time taken to come to rest is 2 min. and the average speed is 120 revs. per min. Hence angle turned through $= 120\times 2\times 2\pi = 1{,}508$ radians.

§ **6. Relative Motion.** Two parts of a machine, each of which has motion relative to a fixed part, say the frame of the machine or the earth, have, in general, motion relative

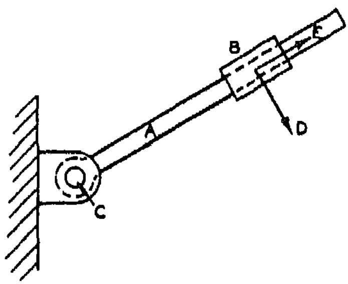


Fig. 2.

to each other, or relative motion. As an example of relative motion consider a swinging link A, Fig. 2, which can rotate about the fixed centre C; a block B is capable of sliding along the link. If the link A is considered for the moment fixed, the block B simply slides along A. Let the direction of the sliding be indicated by the arrow E. Now consider the case when A is moving in the direction of the arrow D, while B is sliding along A. The actual motion of B relative to the fixed frame is intermediate between the arrows E and D,

but the motion of B relative to A is still in the direction of the arrow E. Thus the motion of B relative to A is not affected by any motion of A.

In general, the relative motion between two bodies is not affected by any motion they have in common. This may be exemplified by considering the motion of the crosshead of a locomotive relative to the frame. This relative motion is not altered by the motion of the locomotive along the rails or when rounding a curve.

§ 7. **Use of Vectors.** As already explained, the complete specification of velocity includes the statement of the direction in which the motion is taking place. Quantities which are related to definite directions in space are termed Vector Quantities and may be represented by directed lines called Vectors. Force, acceleration, velocity, electric current are examples of vector quantities, and each of these may be represented by means of vectors.

Problems involving relative motions may be conveniently solved by the use of vectors, and although other methods may be used, the student is advised to make himself familiar with the vector treatment of the solution of various kinds of problems.

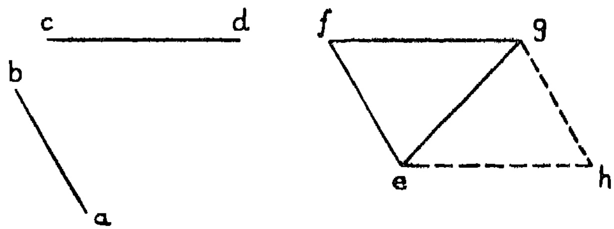
Fig. 3.

§ 8. **Notation of Vectors.** In Fig. 3, a line *ab* is shown. When referred to as a line, *ab* or *ba* may be used. As a vector, *ab* indicates that the motion of a body B relative to a body A is parallel to *ab* and that the direction of the motion is from *a* to *b*. The length of the vector *ab* may conveniently represent the magnitude of the velocity of B relative to A. Referring to Fig. 2, B is assumed to slide along A in the direction of the arrow E; then if *ab*, Fig. 3, is drawn parallel to the arrow E, Fig. 2, and is made of such a length as to represent

the magnitude of the velocity of B relative to A, the vector ab represents completely the motion or velocity of B relative to A.

Conversely, the vector ba indicates that the velocity of A relative to B is parallel to ba and in a direction from b to a. It is thus seen that a vector may completely represent a velocity (or acceleration) in that it represents the magnitude, the path, and the direction.

§ 9. **Addition of Vectors.** The addition of vectors is easily accomplished by simple geometrical construction. In Fig. 4 (a), let ab and cd represent two vectors whose sum is



(a) Fig. 4. (b)

required. In Fig. 4 (b), draw ef equal and parallel to ab and fg equal and parallel to cd; then the vector eg, found by joining e and g, is the vector sum of ef and fg and consequently represents the vector sum of ab and cd. The vectors may be added in any order, since eh could be drawn to represent cd and hg to represent ab; again the vector sum of eh and hg is eg. The triangle efg is known as a vector diagram which may be interpreted by a vector equation: thus $eg = ef + fg$. Similarly, $eg = eh + hg$.

Any number of vectors may be added together by treating the vectors in pairs. The rule for adding vectors may be stated thus: draw the first vector; at the end of the first vector place the beginning of the second vector; at the end of the second vector place the beginning of the third vector; proceeding in this manner until the last vector is in position, the sum of the vectors is represented by the line joining the beginning of the first vector to the end of the last vector.

§ 10. **Subtraction of Vectors.** Referring to Fig. 3, it has been seen that the vectors ab and ba have two things in

common, i.e. magnitude and path, and that the vector *ab* indicates that the motion is from *a* to *b*, while the vector *ba* indicates that the motion is from *b* to *a*. The vector *ab* is thus equal to minus vector *ba*. The subtraction of two vectors *ab* and *cd*, Fig. 4 (*a*), is accomplished by adding the vectors *ab* and $-cd$, or by adding *ab* and *dc*. In Fig. 5, *ef* is drawn equal and parallel to *ab*, *fg* equal and parallel to *dc* (i.e. $-cd$);

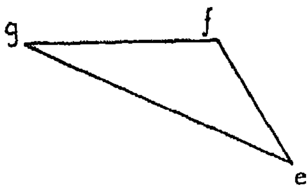


Fig. 5.

the difference of the vectors *ab* and *cd*, Fig. 4 (*a*), is found by joining *e* to *g* as shown in Fig. 5. The triangle *efg* is known as a vector triangle in which

$$\begin{aligned} eg &= ef+fg \\ &= ab+dc \\ &= ab-cd. \end{aligned}$$

§ **11. Interpretation of Vector Triangle.** Considering the vector triangle *efg*, Fig. 4 (*b*), it has been seen that

$$eg = ef+fg.$$

This may be written $eg = fg+ef$.

Now *eg* may represent the velocity of a body *G* relative to another body *E*, *fg* the velocity of *G* relative to a third body *F*, and *ef* the velocity of *F* relative to *E*. The above equation may thus be expressed in words:

Velocity of *G* relative to *E* = velocity of *G* relative to *F* + velocity of *F* relative to *E*.

The method of notation is of fundamental importance in the use of vectors. Following the usual convention, capital letters *E*, *F*, and *G* refer to the actual parts of a machine or to different moving bodies having relative motion, small letters *e*, *f*, and *g* refer to the corresponding vectors. Other letters may be substituted for *E*, *F*, and *G* and the general

form of this important fundamental velocity equation may be written:

Velocity of A relative to C = velocity of A relative to B
+velocity of B relative to C.

The interpretation of this general equation is

$$ca = ba + cb,$$

which can be written

$$ca = cb + ba.$$

This form of the equation will enable the student to construct the vector triangle more readily. On the left side we have ca; on the right side c is the commencing letter and a the finishing letter.

EXAMPLE 6. Two motor-cars A and B travelling along cross-roads pass each other at the junction. A is going at 15 miles per hour in a direction 120° east of north; B is going at 22·5 miles per hour in a direction 10° north of east. How far and in what direction are they apart at the end of one minute? What is the velocity of B relative to A?

Forming the fundamental vector equation and assuming the earth (E) stationary, we have:

Velocity of B relative to A = velocity of B relative to E
+ velocity of E relative to A.

Expressing this in vector quantities:

$$ab = eb + ae = ae + eb.$$

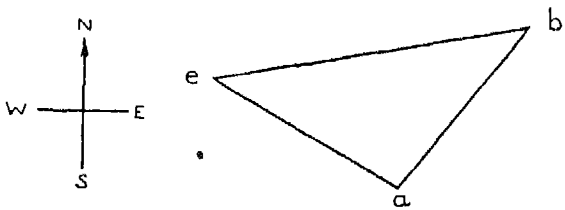


Fig. 6.

In Fig. 6, ae is drawn parallel to the direction 120° east of north and is made 15 units long to represent the velocity of the earth (E) relative to A. The vector eb is drawn 10° north of east and is made 22·5 units long to represent the velocity of B relative to the earth E. The vector ab represents the velocity of B relative to A and this scales 14·6 units, representing a velocity of 14·6 miles

per hour. The direction of ab is 51° north of east. The distance apart at the end of one minute is $\frac{14\cdot6}{60}$ or 0·243 mile.

EXAMPLE 7. A motorist is travelling due north at 30 miles per hour when the wind is blowing from the north-east at 15 miles per hour. What is the direction and speed of the wind as it appears to the motorist? [*Inst. C. E.*]

Let A represent the motorist, E the earth, and W the wind.

Velocity of W relative to A = velocity of W relative to E + velocity of E relative to A,

or $$aw = ew + ae = ae + ew.$$

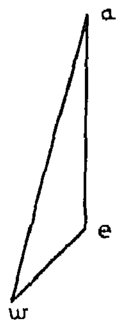


Fig. 7.

In Fig. 7, ae is drawn due south and is made 30 units long to represent the velocity of the earth (E) relative to the motorist (A). The vector ew is drawn as coming from north-east and is made 15 units long to represent the velocity of the wind (W) relative to the earth (E). The vector aw represents the velocity of W relative to A; this measures 42·2 units, representing 42·2 miles per hour. The angle eaw measures 14·5°, giving the apparent direction of the wind to the motorist as 14·5° east of north.

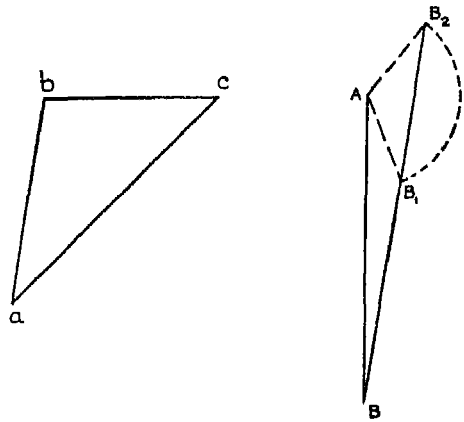
EXAMPLE 8. At midnight a vessel A was 40 miles due north of a vessel B; the vessel A was steaming 20 miles per hour on a south-west course, and B 12 miles per hour due west. They could exchange signals when 12 miles apart. When could they commence signalling, and how long could they continue?

Let C represent the sea.

Velocity of B relative to A = velocity of B relative to C + velocity of C relative to A,

or $$ab = cb + ac = ac + cb.$$

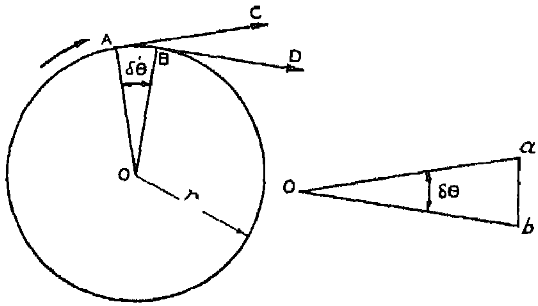
In Fig. 8 (a), ac is drawn to represent the velocity of C relative to A and is made 20 units long; cb is drawn to represent the velocity of B relative to C and is drawn 12 units long; then ab represents the velocity of B relative to A; this measures 14·3 units, representing a velocity of 14·3 miles per hour. The magnitude and direction of the velocity of B relative to A are now known and A can be regarded as stationary, B as steaming in a direction from a to b at 14·3 miles per hour.



(a) Fig. 8. (b)

In Fig. 8 (b), A represents the position of the vessel A at midnight and B the position of the vessel B 40 miles due south. AB is made 40 units long. Note that the scale in (b) is different from that in (a), and that (b) is a displacement diagram whereas (a) is a velocity diagram. From B a line BB_2 is drawn to represent the path and direction in which B is steaming relative to A. Since they can signal when 12 miles apart, a circular arc B_1B_2 is drawn with A as centre and radius equal to 12 units. B_1 represents the position of B when they can commence signalling and B_2 the position when they cease signalling. BB_1 is the distance travelled since midnight until signalling commences; this distance is 29 miles; thus the time-interval is $\frac{29}{14\cdot3}$ or 2·03 hours. The distance B_1B_2 represents the distance travelled while the two vessels can continue to signal; this distance measures 21 miles and the time-interval during which they can signal is $\frac{21}{14\cdot3}$ or 1·47 hours.

§ **12. Centripetal or Radial Acceleration.** When a particle is moving in a circular path with constant angular velocity, the peripheral or linear speed is constant but the direction is continually changing as the particle moves round. In Fig. 9 (*a*), consider a particle moving round a circular path of radius r and whose fixed centre is O. Let the constant angular velocity be ω radians per sec. At a particular instant let the particle be at A. After a small interval of time, δt sec., the particle has moved to a position B and has moved through an angle $\delta\theta$.



(*a*) Fig. 9. (*b*)

At A the particle is moving instantaneously perpendicular to OA as shown by the arrow C. At B the particle is similarly moving instantaneously perpendicular to OB as shown by the arrow D. The two velocities are equal in magnitude, each being equal to $r\omega$, but the direction of the velocity at B is different from that at A; hence, some velocity has been added to the velocity at A to produce the changed (in direction only) velocity at B. To find this added velocity we can construct a vector diagram and find the velocity of B relative to A.

Velocity of B relative to A = velocity of B relative to O + velocity of O relative to A,

or $$ab = ob + ao = ao + ob.$$

In Fig. 9 (*b*), ao is drawn equal to $r\omega$ and parallel to CA to represent the velocity of O relative to A. The vector ob is drawn also equal to $r\omega$ and parallel to the arrow D to represent the velocity of B relative to O; then ab represents the velocity of B relative to A, i.e. the change of velocity or the

velocity that has been added to that at A to produce the velocity $r\omega$ perpendicular to OB at B.

In the vector triangle oab, Fig. 9 (b), the angle $\delta\theta$ may be made as small as desired, since the time-interval δt may be chosen very small; when the angle $\delta\theta$ is very small, then $ab = oa.\delta\theta$ or $ab = r.\omega.\delta\theta$.

The vector ab represents the change of velocity that has taken place in time δt, and hence the rate of change of velocity is $r.\omega.\frac{\delta\theta}{\delta t}$. Rate of change of velocity (with respect to time) is acceleration and $\frac{\delta\theta}{\delta t}$ is the rate of change of angular displacement with respect to time; this latter quantity thus becomes equal to ω, the angular velocity, and the acceleration thus becomes $r\omega.\omega$ or $r.\omega^2$ in a direction from a to b. When A and B, Fig. 9 (a), are chosen very close together, ab, Fig. 9 (b), becomes radial and the acceleration is known as radial or centripetal acceleration.

Let $\qquad a_c =$ centripetal acceleration,

$$a_c = r\omega^2. \tag{7}$$

Example 9. A particle is moving in a circular path of 15-in. radius with an angular velocity of 20 radians per sec. Determine the centripetal acceleration.

Using equation (7) above and expressing r in feet:

$$a_c = \tfrac{15}{12} \times 20^2 = 500 \text{ ft. per sec. per sec.}$$

§ **13. Resultant Acceleration.** When the angular velocity of a particle moving in a circular path is not constant, but is subject to an angular acceleration (with consequent peripheral or linear acceleration), the resultant acceleration may be found by adding the vectors representing the centripetal and peripheral accelerations. Since the centripetal acceleration is always radial and perpendicular to the instantaneous direction of motion and the peripheral acceleration is perpendicular to the radius, these two accelerations are always mutually perpendicular, and the resultant acceleration is easily found by applying the well-known relationship which exists between the sides of a right-angled triangle, that is, by

extracting the square root of the sum of the squares of the respective accelerations.

EXAMPLE 10. A motor-car passes round a curve of 100 ft. radius and at a given instant has a speed of 20 miles per hour. The car is accelerating at the rate of 10 miles per hour in 3 sec. Find the resultant acceleration.

Centripetal acceleration $= r\omega^2$, which may be written $\frac{v^2}{r}$, since $v = r\omega$.

$$a_c = \left(\frac{20\times 5{,}280}{60\times 60}\right)^2 \times \frac{1}{100} = 8{\cdot}61 \text{ ft. per sec. per sec.}$$

$$\text{Peripheral acceleration } a = \frac{10\times 5{,}280}{60\times 60\times 3} = 4{\cdot}89 \text{ ft. per sec. per sec.}$$

Resultant acceleration $= \sqrt{(8{\cdot}61^2+4{\cdot}89^2)} = 9{\cdot}9$ ft. per sec. per sec.

§ 14. **Simple Harmonic Motion.** Consider a particle moving in a circular path of radius r and constant angular

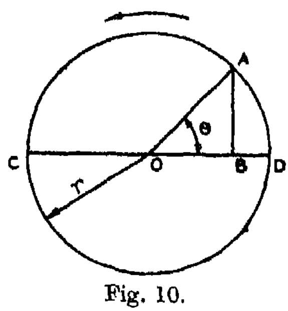


Fig. 10.

velocity ω radians per sec. In Fig. 10, at any particular instant suppose the particle is at A. If any diameter such as CD is chosen and a line AB is drawn perpendicular to CD, then the position of B on CD alters as A moves in its circular path. The motion of B is said to be periodic or simple harmonic.

The radial acceleration of A is $r\omega^2$ or $\frac{v^2}{r}$ in a radial direction from A to O. The acceleration of B is the component of this radial acceleration projected on to CD and in magnitude is $r\omega^2\cos\theta$, where θ is the angular displacement of A from the diameter CD in the direction of rotation. The acceleration of

B varies as $r \cos\theta$ or OB, hence the acceleration varies directly as the distance from the centre position O.

While A makes one complete revolution, B moves through O to C, thence to D, and finally to B; this time is called the period of a complete vibration. The distance travelled by A in one complete revolution is $2\pi r$, hence the time taken is the circumference divided by the velocity.

Let t = periodic time,

then $$t = \frac{2\pi r}{r\omega} = \frac{2\pi}{\omega}.$$

In cases where t is very small, that is, the number of vibrations per second is large, it is usual to give the frequency $\frac{\omega}{2\pi}$ instead of the periodic time.

The amplitude is half the motion of B, i.e. the radius of the circular path, or half the maximum travel.

It should be noted that as B passes through O, the acceleration is zero and B has its maximum velocity; when B reaches C or D, the acceleration is a maximum of value $r\omega^2$ and the velocity is zero.

EXERCISES. I

1. The speed of a train increases uniformly from 15 to 40 miles per hour in 75 sec. What is the acceleration?

2. A train reduces its speed at a constant rate from 50 to 20 miles per hour in 2 min. Determine the linear retardation of the train and the angular retardation of the driving-wheels of the locomotive, which are $6\frac{1}{2}$ ft. in diameter.

3. A centrifugal pump impeller is $2\frac{1}{2}$ ft. in diameter and has a speed of 300 revs. per min. The water leaves the impeller with an absolute velocity of 5 ft. per sec. radially. Find the velocity and direction of the water relative to the impeller.

4. A steamer, which can travel 15 miles per hour in still water, has to go 30 miles in a direction 40° east of north, in a steady southerly current of 3 miles per hour. In what direction must the steamer be steered, and how long will it take to complete the journey? [*I. Mech. E.*]

5. The tracks of two ships cross at a point A. One ship is 20 nautical miles east of A, steaming west at 15 knots; the other ship is 10 nautical miles south of A, steaming north at 12 knots. Find the velocity of the first relative to the second, and find how near they approach each other. [*I. Mech. E.*]

6. A body is observed to travel 10 ft. in 2 sec. and 60 ft. in the next 5 sec., its motion being in a straight line with uniform acceleration. Find that acceleration, and the initial velocity, and how far it travels in 30 sec. from the beginning of the above observations.

[*I. Mech. E.*]

7. Calculate the uniform retardation of a train which is brought to rest from a speed of 60 miles per hour in a mile and a half.

[*Inst. C. E.*]

8. A body moving along a straight line with uniform acceleration passes three points A, B, C, and it is observed that the body passes B 3 sec. and C 5 sec. after it has passed A. If the distances AB and BC are 15 and 22 ft. respectively, find the acceleration of the body and its speed when passing the point A. [*Inst. C. E.*]

9. An aeroplane in flight in a steady wind which blows at 40 miles per hour in a north-east direction is propelled at a speed of 120 miles per hour relative to the wind. In what direction must the pilot apparently steer to maintain a course due west?

What is his speed in the direction due west?

10. A train is going south at 26 miles per hour. Later its velocity is 37 miles per hour in a direction 52° north of east. What has been the change of velocity?

11. A vessel travelling due east at a speed of 10 knots enters a current which is flowing due north-east. To maintain its course the vessel is headed slightly south of east and its speed is then found to be 11 knots. Find the speed of the current and the angle south of east to which the vessel is directed. 1 knot = 6,080 ft. per hour.

12. A patrol boat which can make 30 knots sights a ship due north travelling 10° north of west at 20 knots and wishes to intercept it. Find the direction in which it must go to reach it at the earliest possible moment. [*I. Mech. E.*]

13. From a balloon ascending with a velocity of 32 ft. per sec. a stone is freed; it reaches the ground in 17 sec. How high was the balloon when the stone was released? [*Inst. C. E.*]

14. A cyclist 'free wheels' down a hill half a mile long having a slope of 1 in 25. At the top of the hill his speed is 10 miles per hour, and he traverses the whole length of the hill in $1\frac{1}{2}$ min. Calculate the uniform acceleration.

15. An airship at the moment it drops a bomb is travelling at 48 ft. per sec. along an upward path inclined at 30° to the horizontal; it is 200 ft. above the ground. Find how far the point at which the bomb reaches the ground will be from the point perpendicularly below that at which the bomb was released. Neglect air friction.

[*Inst. C. E.*]

16. A flagship is steaming south-east at 14 knots. A cruiser, speed 20 knots, is 10 sea-miles on the port bow (i.e. north-east from the

flagship). The cruiser is ordered to take station 5 sea-miles ahead of the flagship. What course should she steer, what time will she take, and how many sea-miles will she steam? [*Inst. C. E.*]

17. A man who can row at 3 miles per hour in still water rows across a stream running at 2 miles per hour with the nose of the boat pointing upstream at an angle of 45° with the bank. What is the true velocity of the boat, and how long will it take the man to cross the stream, which is 15 yds. wide?

18. A body projected vertically upwards attains a maximum height of 1,520 ft. Calculate the velocity of projection and the complete time of flight in the air. At what altitude would this body meet a second body projected vertically 4 sec. later with a speed of 400 ft. per sec.?

19. At noon A is 10 miles due east of B, and is travelling at 14 miles per hour in a direction 30° west of north. B is travelling north-east at 10 miles per hour. Find the velocity of A relative to B, and find the least distance apart and the time at which this occurs. [*I. Mech. E.*]

20. A stone is dropped from the top of a cliff 400 ft. high, and 1 sec. afterwards another stone is thrown down and strikes the first stone when it has just reached the foot of the cliff. Find the speed with which the second stone was thrown, neglecting air resistance. [*I. Mech. E.*]

21. A light rod, 4 ft. long, carrying equal masses at its ends, is let fall with a rotary motion from a cliff 400 ft. high, the velocity of each mass relative to the centre of the rod being 22 ft. per sec. Find how many revolutions it will make during the fall. [*I. Mech. E.*]

22. A ship A steaming due west at 9 knots sights another ship B due south and travelling at a speed of $16\frac{1}{2}$ knots in a direction 30° north of west. After an interval of 30 min. B is due south-west of A. Find: (*a*) what further interval of time must elapse before B is due west of A; (*b*) what is then the distance between the ships; (*c*) the distance between the ships when they are nearest together; (*d*) what time elapses after first sighting before they are nearest together.

23. Determine the magnitude of the acceleration of a point on the circumference of a fly-wheel, 6 ft. 2 in. in diameter, when making 110 revs. per min.

24. Determine the greatest and least magnitudes of the speed and the acceleration of a point on the rim of a 6 ft. 6 in.-diameter driving-wheel of a locomotive travelling at 55 miles per hour.

25. A body moving horizontally with a velocity of 50 ft. per sec. has its velocity changed to the vertical direction and to 100 ft. per sec. in 4 sec. Find the *mean acceleration*.

CHAPTER II

FORCE, WORK, AND ENERGY

§ 15. **Force.** Force is often defined as that which tends to produce, produces, or alters the motion of a body. Enlarging upon this definition, a small child pushing against a table probably fails to move the table, but is nevertheless exerting a force. An adult pushing against the table will probably cause the table to move and is thus exerting a force. A batsman hitting a cricket-ball alters the motion of the ball by means of an impulsive force.

To specify completely a force, its magnitude, direction, and a point of application must be known. A force, having direction and magnitude, may be represented by a vector in the same way that velocity and acceleration may be represented.

§ 16. **Weight and Mass.** The weight of a body may be defined as the force with which the earth attracts a body, and since this force varies slightly over the surface of the earth, the weight varies correspondingly. The mass of a body is constant at all points on the surface of the earth and may be defined as the amount of stuff or matter contained in the body. The British standard of mass is a piece of platinum which is in the keeping of the Board of Trade, London.

The force which the earth exerts on this standard mass is called the pound, and the weight of this mass is regarded as one pound in London.

§ 17. **Momentum.** The momentum or quantity of motion a body possesses is measured by the product of the mass and its velocity. If the velocity changes in magnitude or direction, then the momentum changes accordingly.

§ 18. **Newton's Laws of Motion.** These laws are of fundamental importance and may be stated thus:

(i) Every body continues in a state of rest or of uniform motion in a straight line except in so far as it may be compelled to change that state by means of a force.

(ii) The rate of change of momentum is proportional to the

impressed force and takes place in the direction in which the force acts.

(iii) To every action there is always an equal and opposite reaction.

The second law of motion induces a most important dynamical equation which may be deduced thus:

Let P = force acting in pounds,

W = weight of body in pounds,

g = acceleration due to gravity,

M = mass.

According to Newton's second law:

$P \propto$ rate of change of momentum, i.e. $P \propto$ rate of change of $M \times v$.

Assuming the mass M to remain constant, the variable in $M \times v$ is v; hence

$$P \propto M \times \text{rate of change of } v.$$

But rate of change of v with respect to time is acceleration, then

$$P \propto M \times a, \text{ where } a = \text{acceleration}.$$

This may be written in the form of an equation:

thus $$P = c \,.\, M \,.\, a, \text{ where } c \text{ is a constant}.$$

The unit of mass has not yet been defined, and accordingly, if the unit of mass is chosen such that unit force acting on unit mass produces unit acceleration, the value of the constant c becomes unity and the above equation becomes

$$P = M \,.\, a.$$

When a body is allowed to fall freely the force acting on the body is its weight vertically downwards; this produces an acceleration of g ft. per sec. per sec. Substituting the value of W for P and g for a in the above equation, we get

$$W = M \,.\, g$$

or $$M = \frac{W}{g}.$$

Thus in engineers' units the mass of a body may be regarded

as its weight in pounds divided by g. The dynamical equation now becomes

$$P = \frac{W}{g} . a. \tag{1}$$

EXAMPLE 1. A force of 50 lb. acts on a moving mass and changes its velocity from 30 to 15 miles per hour in 10 min. Find the weight of the mass.

$$\text{Acceleration} = \frac{15 \times 5{,}280}{60 \times 60 \times 10 \times 60} = \frac{11}{300} \text{ ft. per sec. per sec.}$$

Using equation (1): $50 = \frac{W}{32{\cdot}2} \times \frac{11}{300}$.

$$\therefore\ W = \frac{50 \times 32{\cdot}2 \times 300}{11 \times 2{,}240} = 19{\cdot}6 \text{ tons.}$$

§ **19. Impulse.** A force may act upon a body steadily or it may act for a very short interval of time. When the time is exceedingly short the force is referred to as an impulsive force or blow. The magnitude of the blow is measured by the product of the mass and the acceleration or change of momentum divided by the time-interval.

EXAMPLE 2. A tup weighing 500 lb. is allowed to fall freely through a distance of 3 ft. and is brought to rest in 0·01 sec. Find the average force of the blow.

Velocity of tup on striking $= \sqrt{(2g \times 3)} = 13{\cdot}9$ ft. per sec.

$$\text{Blow} = \frac{W}{g} \times \frac{\text{change of velocity}}{\text{time}} = \frac{500}{32{\cdot}2} \times \frac{13{\cdot}9}{0{\cdot}01} = 21{,}600 \text{ lb.}$$
$$= 9{\cdot}64 \text{ tons.}$$

§ **20. Centrifugal Force.** When a particle is moving in a circular path of radius r at a constant angular velocity of ω radians per sec., it has a radial or centripetal acceleration of magnitude $r\omega^2$. If the particle weighs W lb. the force required to cause this acceleration is $\frac{W}{g} r\omega^2$ or $\frac{W}{g}\frac{v^2}{r}$ lb. This force is in tons if W is in tons. Consider the case of a small stone at the end of a piece of string, moving in a circular path. The stone is constrained to move in its circular path by the tension in the string, this tension being equal to $\frac{W}{g} r\omega^2$. This tension or force is called centripetal force and acts radially inwards. When a body is being accelerated there is a reluctance or

resistance to acceleration exactly equal and opposite to the accelerating force. Consequently, in the case of a rotating mass there is a resistance to acceleration acting outwards exactly equal and opposite to the centripetal force. This *outward* resistance is termed centrifugal force, and this in turn is transmitted to the axis about which rotation is taking place.

Hence centrifugal force $= \dfrac{W}{g} r\omega^2$.

This centrifugal force becomes very important when a number of masses are rotating about an axis at high speed. Each of the masses (assuming the centre of gravity of each mass does not coincide with the axis) will have a disturbing influence on the axis and in practice it is usual to attempt to balance these forces.

§ **21. Torque.** The effect of a force depends very much upon its point of application. Take the case of a lever capable of rotating round the axis of a spindle. If a force is applied to the lever such that the force passes through the axis of the spindle, no apparent effect is produced on the lever; if, however, the force is applied to the free end of the lever, a movement of the lever about the spindle may occur. The effect of the force is a turning effect and the product of the magnitude of the force and the perpendicular distance to the centre of the spindle is called the turning-moment or torque. If the force is in pounds and the distance in feet, the torque is in foot-pounds or pounds-feet. The latter term is more generally used to distinguish between the units of work and energy.

§ **22. Moment of Inertia.** A body, free to rotate about an axis, may be caused to rotate by the application of a torque. Consider a small mass of weight W attached to a light rod the mass of which may be neglected. In Fig. 11, let A represent the small mass, which can rotate in a circular path of radius r. Let P be the magnitude of the force which causes rotation.

Let $a =$ peripheral or linear acceleration,

$\alpha =$ angular acceleration,

then $$P = \frac{W}{g} \cdot a = \frac{W}{g} r\alpha$$

and $$P.r = \frac{W}{g}.r\alpha.r = \frac{W}{g}r^2\alpha.$$

But $P.r$ is the torque and may be designated by T:

then $$T = \frac{W}{g}r^2.\alpha.$$

The term $\frac{W}{g}r^2$ is the second moment of the mass and occurs so frequently in engineering problems that it is given a special symbol I, where $I = \frac{W}{g}r^2$. I is often referred to as the moment of inertia.

Thus $$T = I.\alpha. \tag{2}$$

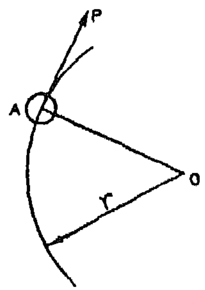


Fig. 11.

The effect of a torque is to produce or tend to produce an angular acceleration, analogous to the effect of a force, which tends to produce linear acceleration.

In the case of a larger body rotating about an axis the moment of inertia may be found by taking a series of small masses w_1, w_2, w_3, etc., at radii r_1, r_2, r_3, etc. The moment of inertia of the whole mass is found by adding up the individual moments of inertia of the small masses.

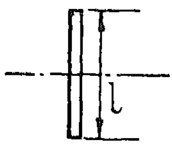
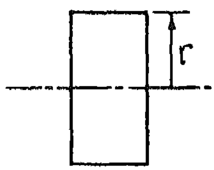
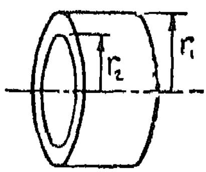
Thus $$I = \frac{w_1}{g}r_1^2 + \frac{w_2}{g}r_2^2 + \frac{w_3}{g}r_3^2 + \text{etc.}$$

The summation of these quantities for regular shaped masses is conveniently accomplished by integration.

In a general manner the moment of inertia of a body may be written $I = \frac{W}{g}k^2$, where k is known as the radius of gyration

and is that radius at which the mass may be considered as concentrated for the purpose of calculating the moment of inertia.

Values of the moment of inertia for various masses likely to be found in practice are given in the following table:

Description.	*Illustration*	*Moment of inertia* I	*Radius of gyration* k
Small mass W at radius r		$\frac{W}{g} r^2$	r
Thin rod of weight W, axis passing through the centre and perpendicular to the rod		$\frac{W}{g} \cdot \frac{l^2}{12}$	$\frac{l}{\sqrt{12}}$
Solid cylinder of weight W		$\frac{W}{g} \cdot \frac{r^2}{2}$	$\frac{r}{\sqrt{2}}$
Hollow cylinder of weight W		$\frac{W}{g} \cdot \frac{r_1^2 + r_2^2}{2}$	$\sqrt{\frac{r_1^2 + r_2^2}{2}}$

§ **23. Effects of Force.** Considering the effect of force on a movable body, it has been seen that a force may be utilized to give the body acceleration; a second effect is that when motion takes place, some force must be utilized in overcoming frictional resistances; thirdly, if the body is moving up or down an incline, some force is required to lift or to support the body. Thus the total force acting on a body is, in general, the sum of the three forces required to

accelerate, to overcome frictional resistances, and to lift the body.

The force required to accelerate a body is readily found from the dynamical equation $P = \frac{W}{g} . a$. In the case of rolling bodies, the frictional resistance is usually expressed in pounds per ton. Thus a truck weighing 10 tons and having a frictional resistance of 15 lb. per ton requires a force of 10×15 or 150 lb. to cause steady motion on the level. If the truck is on an incline of, say, 1 in 20, the force parallel to the plane to support the truck is equal in magnitude to the component along the plane of the weight of the truck, i.e.

$$\frac{10 \times 2{,}240}{120} = 186{\cdot}7 \text{ lb.}$$

The total force producing motion may be stated in the form of an equation:

Force producing motion = force to accelerate + force to overcome frictional resistances + force to support the body.

Of the forces to accelerate and to support the body, either or both of these may be zero. The force to accelerate is zero when the body has steady motion and the force to support may be zero when the body is on the level.

Example 3. The draw-bar pull on a train weighing 300 tons is 5 tons. Taking the frictional resistances to be 15 lb. per ton, find the acceleration with which the train will move up an incline of 1 in 200.

Force to support $= \frac{300}{200} \times 2{,}240 = 3{,}360$ lb.

Force to overcome frictional resistances $= 300 \times 15 = 4{,}500$ lb.

Applying the above equation of force,

$$5 \times 2{,}240 = \text{force to accelerate} + 4{,}500 + 3{,}360,$$

$$\therefore \text{ force to accelerate} = 11{,}200 - 7{,}860 = 3{,}340 \text{ lb.}$$

The acceleration is found by using the dynamical equation

$$P = \frac{W}{g} . a \quad \text{or} \quad a = \frac{3{,}340 \times 32{\cdot}2}{300 \times 2{,}240}$$

$$= 0{\cdot}16 \text{ ft. per sec. per sec.}$$

§ 24. **Effects of Torque.** Torque for circular motion is analogous to force for linear motion and the total torque

acting on a body capable of rotation about a fixed axis may be utilized in accelerating the whole system, overcoming frictional resistances, and in supporting the mass.

EXAMPLE 4. A train of loaded trucks, weighing 20 tons, is drawn up an incline of 1 in 40 by means of a rope coiled round the drum of a winding engine at the top of the incline. The rope is parallel to the incline, the drum weighs 3 tons, its diameter is 7 ft., and its radius of gyration 3 ft. If a constant torque of 7,000 lb.-ft. is applied to the shaft of the drum, find (*a*) the tension in the rope, (*b*) the acceleration of the trucks, (*c*) the speed at which they are travelling at the end of 4 min. Assume the resistance to motion of the trucks as 20 lb. per ton, and neglect the friction at the drum-shaft and the effect of the stiffness of the rope. [*Lond. B.Sc.*]

Torque to accelerate drum

$$= I\alpha = \frac{W}{g}k^2.\alpha = \frac{3\times 2{,}240}{32{\cdot}2}\times 9.\alpha \text{ lb.-ft.}$$
$$= 1{,}878\alpha.$$

Torque to accelerate trucks = force to accelerate × radius of drum

$$= \frac{20\times 2{,}240}{32{\cdot}2}\times a\times 3{\cdot}5 = 4{,}870a \text{ lb.-ft.}$$

Torque to overcome frictional resistances

$$= 20\times 20\times 3{\cdot}5 = 1{,}400 \text{ lb.-ft.}$$

$$\text{Torque to support trucks} = \frac{20\times 2{,}240}{40}\times 3{\cdot}5 = 3{,}920 \text{ lb.-ft.}$$

Applying the equation of torque,

$$7{,}000 = 1{,}878\alpha + 4{,}870a + 1{,}400 + 3{,}920.$$

Substituting $a = r\alpha$,

$$7{,}000 = 1{,}878\,.\,\frac{a}{3{\cdot}5} + 4{,}870a + 5{,}320,$$

$$\therefore\; a = 0{\cdot}311 \text{ ft. per sec. per sec.}$$

The tension in the rope is found by using the equation of force
P = force to accelerate + force to overcome frictional resistances + force to support

$$= \frac{20\times 2{,}240}{32{\cdot}2}\times 0{\cdot}311 + 20\times 20 + \frac{20\times 2{,}240}{40}$$
$$= 1{,}953 \text{ lb.}$$

Speed at the end of 4 min., assuming they start from rest, is found by using one of the equations of motion:

$$v = v_0 + at = 0 + 0{\cdot}311 \times 240 = 74{\cdot}7 \text{ ft. per sec.}$$

§ 25. **Work.** When a force moves through a distance work is done. Since force is required to cause the motion of a body, the work done depends upon the magnitude of the force. The work done also depends upon the distance through which the force acts. If the force is acting in the direction of motion the work done is the product of the force and the distance moved, assuming the force to remain constant.

Work can be represented graphically by plotting the force vertically and the distance moved horizontally. When the force is constant the diagram thus obtained is a rectangle, the area of which represents the work done. In the case of a variable force, the force may be regarded as sensibly constant for an exceedingly short distance, and the work done over this short distance is the product of the force and the short distance. Proceeding in this manner, the whole of the work done can be found by adding up the work done for all the small distances.

If represented graphically, the area under the curve obtained, when force is plotted against distance, represents the work done.

§ 26. **Work done by Torque.** A force P, acting at a radius r from a fixed axis, moves through a distance $2\pi r$ for one complete revolution. The work done per revolution is $P \times 2\pi r$, which may be written $T \times 2\pi$, since T (torque) $= P \times r$. The angle turned through in one revolution is 2π radians, and hence the more general statement may be made that work done by a torque is equal to the product of the torque and the angle turned through in radians.

Let θ = angle turned through in radians:

then $$\text{work done} = T \times \theta.$$

If P is acting on a small mass and the mass makes N revs. per minute, the work done per minute is $T \times 2\pi N$, more usually written $2\pi NT$.

§ **27. Horse-power.** Power is the rate of doing work; the unit of power in Britain is the horse-power, which is defined as 550 ft.-lb. of work per second or 33,000 ft.-lb. per minute. The horse-power of a machine or any other agent is found by dividing the work done in ft.-lb. per minute by 33,000. The horse-power developed by a torque T lb.-ft. making N revs. per min. is $\frac{2\pi NT}{33{,}000}$.

The unit of horse-power used on the Continent is 4,500 kilogram-metres per min., which corresponds to 32,500 ft.-lb. per min.; the Continental horse-power is thus 0·986 of the British horse-power.

The electrical unit of power is the watt, which is a current of 1 ampere at a pressure of 1 volt.

The relation between British horse-power and the watt is 1 British horse-power = 746 watts.

A unit of work frequently used by engineers is the horse-power-hour. This represents one horse-power over a period of 1 hour and is equivalent to $60 \times 33{,}000$ or 1,980,000 ft.-lb. The unit of work used in electrical power is the kilowatt-hour or Board of Trade unit. A kilowatt-hour is 1,000 watts for a period of 1 hour. One Board of Trade unit is thus equivalent to $\frac{1000}{746}$ or 1·34 horse-power-hours.

§ **28. Simple Machines.** Without the necessity of entering into a detailed discussion of any particular machine, it is convenient, at this stage, to introduce the question of simple machines and to consider some of the terms employed when testing these machines.

Regarding a machine as an agent for transmitting or modifying energy, the problem resolves itself into the question of the relation between energy supplied to and energy obtained from the machine. Considering, say, a simple lifting machine, the mechanism of which, for the moment, need not be considered, it consists of a load W to be lifted by means of an effort E, as in Fig. 12. In general, E may be smaller than W. By means of the effort E the load W may be lifted, and by supplying the effort E a definite amount of work is supplied

to the machine in a given time; by lifting the load W in the same time-interval a definite amount of work is obtained from the machine.

§ 29. **Velocity-Ratio and Efficiency.** In the case of a machine in which the distance moved through by the effort bears a constant ratio to the distance moved through by the load, this ratio is known as the velocity-ratio. The term distance-ratio is perhaps a truer interpretation of this ratio, but the former term is in more general use and will be used here.

Let c = distance moved by effort E in a given time,

d = distance moved by load W in the same time,

V = velocity-ratio,

then $$V = \frac{c}{d}.$$

The efficiency of a machine is the ratio of the useful work done by it to the work supplied to it. On account of the internal friction of a machine the work done is always less than that supplied.

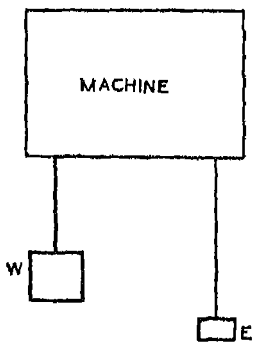


Fig. 12.

Referring to Fig. 12, let the effort E move through a distance c while the load W is lifted a distance d.

$$\text{Work done} = W \times d.$$

$$\text{Work supplied} = E \times c.$$

$$\therefore \text{ Efficiency } \eta = \frac{W \times d}{E \times c} = \frac{W}{E \times \frac{c}{d}} = \frac{W}{E \times V}.$$

Expressed in words, the efficiency is the load divided by the product of the effort and the velocity-ratio.

The ratio of the load W to the effort E is termed the mechanical advantage.

Let M = mechanical advantage.

then $$M = \frac{W}{E} \text{ and } \eta \text{ becomes } \frac{M}{V}.$$

§ **30. Ideal Machine.** In a theoretically perfect machine, that is, one in which there is no internal friction, and the effect of all moving parts of the machine except the load lifted and the effort applied is negligible, the work done by the machine is equal to the work put into it and the efficiency becomes unity or 100 per cent. Such a machine is termed an *ideal machine*.

In an ideal machine the effort required to lift a given load W is less than that in an actual machine.

Let E_1 = effort required in an ideal machine, whose velocity-ratio is V, to lift a load W.

Then, since the efficiency is unity,

$$E_1 = \frac{W}{V}; \ E_1 \text{ is known as the ideal effort.}$$

The actual effort E is greater than E_1 and the difference between these quantities is known as the wasted effort.

Thus, wasted effort $= E - E_1 = E - \frac{W}{V}$.

This wasted effort is simply the effort required to overcome the friction of the machine.

Conversely, a given effort E will operate a greater load on the ideal machine than on an actual machine.

Let W_1 = load lifted on an ideal machine by an effort E,

then $$W_1 = E \times V.$$

If W is the load lifted on the actual machine, $W_1 - W$ represents the loss of load due to the friction of the machine. This loss of load is called friction load.

Thus, friction load $= W_1 - W = E \times V - W$.

Care should be taken when stating the friction of a machine as to whether a wasted effort or friction load is implied.

§ **31. Reversal of Machines.** When the effort on a

machine is removed the load may run backwards or remain stationary. If the load remains stationary the machine is non-reversible or self-locking; if the load runs backwards the machine is said to be reversible. The efficiency of a self-locking machine is less than 50 per cent.

Consider a machine, Fig. 12, in which a load W is raised by an effort E.

Energy supplied to the machine $= E \times c$.

Useful energy $= W \times d$.

$\therefore$ Energy wasted $= E \times c - W \times d$.

If the effort E is now removed, the machine will be reversible if W just runs backwards. The energy wasted in this case in the same time is $W \times d$, and assuming the energy lost in the two cases to be the same:

$$E \times c - W \times d = W \times d$$

or
$$E . c = 2 . W . d.$$

But $\dfrac{W . d}{E . c}$ is the efficiency of the machine when working normally, hence efficiency $= \frac{1}{2}$ or 50 per cent. If the efficiency exceeds 50 per cent. the machine will reverse more readily.

§ **32. Law of a Machine.** In carrying out a test of a simple machine it is usual to observe simultaneous readings between the effort and the load. When these results are plotted on squared paper it is found that the points thus obtained lie very nearly on a straight line. A linear relation between E and W thus exists, and this relation may be expressed in the form of an equation. The equation takes the form

$$E = aW + b,$$

where a and b are constants depending upon the particular machine.

In Fig. 13, AC represents a line drawn through points obtained by plotting values of E vertically against values of W horizontally.

At no load the effort is represented by OA and this clearly represents the constant b, since the effort is b when W is zero, from the equation $E = aW + b$. The constant a is found by calculating the slope of the line and in magnitude is equal to

$\frac{CD}{BD}$, B and C being any two points on the line chosen at random and D the intersection of BD and CD drawn horizontally and vertically respectively. The equation $E = aW + b$ is known as the law of the machine, and by its use it is possible to estimate the effort required to raise a given load.

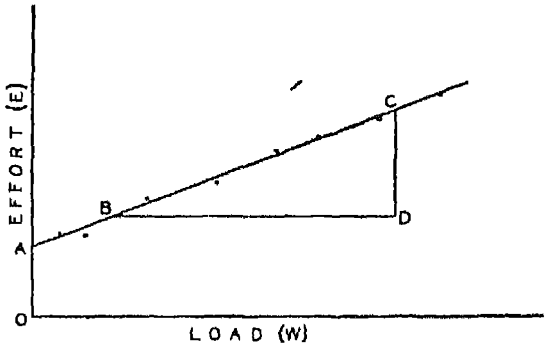


Fig. 13.

EXAMPLE 5. The friction of an unloaded pulley-block is 2·4 lb. The friction increases at the rate of 1·9 lb. per 100 lb. of load lifted by the block. Velocity-ratio, 22. Find the effort required to lift a load of 1,300 lb. and the mechanical efficiency at this load. [*Inst. C. E.*]

Effort required = no load effort + ideal effort + friction effort

$$= 2{\cdot}4 + \frac{1{,}300}{22} + \frac{1{\cdot}9 \times 1{,}300}{100} = 86{\cdot}2 \text{ lb.}$$

$$\text{Efficiency} = \frac{1{,}300}{86{\cdot}2 \times 22} = 0{\cdot}686 \text{ or } 68{\cdot}6 \text{ per cent.}$$

EXAMPLE 6. In a differential pulley-block whose velocity-ratio is 50 it is found that an effort of 13 lb. will lift a load of 400 lb. and 103 lb. will lift 4,000 lb. Find the law of the machine and the efficiency when lifting this latter load.

Let the law be of the form $E = aW + b$.

Substituting values $13 = 400a + b$

$$103 = 4{,}000a + b$$

$$90 = 3{,}600a.$$

$$\therefore\ a = \frac{1}{40}, \text{ whence } b = 3.$$

Law is $$E = \frac{W}{40} + 3.$$

Efficiency $$= \frac{4{,}000}{103 \times 50} = 0{\cdot}776 \text{ or } 77{\cdot}6 \text{ per cent.}$$

§ 33. **Energy.** A body capable of doing work is said to possess energy. In mechanics the two important kinds of energy are potential and kinetic.

A body may possess potential energy due to its position relative to some other body, say the earth or frame of a machine. A mass weighing W lb. at a height H ft. above a given datum possesses $W.H$ ft.-lb. of energy, and by allowing the mass to fall this work can be utilized.

Kinetic energy of a body is that due to its motion. A stationary body has no kinetic energy but may or may not possess potential energy; a moving body possesses kinetic energy but may or may not possess potential energy. When a moving body is brought to rest, its kinetic energy is destroyed and work is done in bringing the body to rest.

Consider a mass of weight W lb. at a height of H ft. above a given datum. The potential energy possessed by the body is WH ft.-lb. If the body is allowed to fall freely it acquires a velocity of $\sqrt{(2gH)}$ when it reaches the datum level, or

$$v = \sqrt{(2gH)}.$$

Expressing H in terms of v, $H = \frac{v^2}{2g}$, and since there has been no change of energy, the energy possessed by the mass when it reaches the datum is still WH or $\frac{Wv^2}{2g}$ ft.-lb. The energy is thus expressed in terms of v and is therefore kinetic energy.

A small mass of weight W, rotating about an axis at constant angular speed ω radians per sec. at a radius of r ft., possesses kinetic energy $\frac{Wv^2}{2g}$ or $\frac{Wr^2\omega^2}{2g}$ ft.-lb. For larger masses the expression for kinetic energy is generally expressed as $\frac{Wk^2\omega^2}{2g}$, where k is the radius of gyration. Using the value $I = \frac{W}{g}k^2$, the kinetic energy of a rotating body becomes $\frac{1}{2}I\omega^2$.

When a body is accelerated, force is required to produce the acceleration and hence work is done. The work done in this manner is equal to the change of kinetic energy in the body. Considering a force P acting on a mass of weight W and producing an acceleration a, the velocity acquired when a distance s has been traversed is found from $v^2 = v_0^2 + 2as$. Multiplying throughout by $\frac{W}{2g}$ and transposing,

$$\frac{W}{2g}v^2 - \frac{W}{2g}v_0^2 = 2\frac{W}{2g}.as = P.s.$$

The left side of the equation represents the change of kinetic energy and the right side the work done.

§ 34. Total Kinetic Energy. A body which has rotational motion in addition to linear motion, such as a rolling wheel, has a total kinetic energy due to its linear motion and to its rotation. Total kinetic energy $= \frac{1}{2}I\omega^2 + \frac{W}{2g}v^2$.

EXAMPLE 7. A railway truck is mounted on two pairs of wheels of 21 in. radius, each pair weighing 10 cwt. and having a radius of gyration of 18 in. The total weight of the truck is 5 tons. Find the kinetic energy at 20 miles per hour.

Total kinetic energy = kinetic energy of rotating wheels + kinetic energy of whole truck

$$= \frac{1}{2}\cdot\frac{1}{32 \cdot 2}\cdot(1\tfrac{1}{2})^2\frac{v^2}{(1\tfrac{3}{4})^2} + \frac{5}{32 \cdot 2\times 2}v^2 = 0 \cdot 089v^2,$$

$$= 0 \cdot 089\times\left(\frac{20\times 5,280}{60\times 60}\right)^2 = 76 \cdot 6 \text{ ft.-tons.}$$

EXAMPLE 8. A fly-wheel alters in speed from 99 to 101 revs. per min. and its kinetic energy alters by 500,000 ft.-lb. Calculate the moment of inertia of the fly-wheel and its kinetic energy at 1 rev. per min. [*Inst. C. E.*]

Let ω_1 = initial angular speed, ω_2 = final angular speed.

Change of kinetic energy $= \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$.

$$\therefore \tfrac{1}{2}I(\omega_2^2 - \omega_1^2) = 500,000.$$

$$I = \frac{2\times 500,000}{\left(\frac{2\pi}{60}\right)^2(101+99)(101-99)} = 228,000 \text{ in lb. and ft. units.}$$

$$\text{Kinetic energy at 1 rev. per min.} = \frac{1}{2} \times 228{,}000 \times \left(\frac{2\pi}{60}\right)^2$$
$$= 1{,}250 \text{ ft.-lb.}$$

EXAMPLE 9. A steam engine develops 80 horse-power at 100 revs. per min. against a steady load. The fly-wheel weighs 3 tons and has a radius of gyration of 5 ft. If the load suddenly changes to one-eighth of the initial value, and there is no change in the steam supply for 2 revs. after the reduction of load, calculate the change of speed from beginning to end of this period.

[*Lond. B.Sc.*]

$$\text{Excess energy developed in 2 revs.} = \frac{80 \times 33{,}000}{100} \times \frac{7}{8} \times 2$$
$$= 46{,}200 \text{ ft.-lb.}$$

This excess energy must be absorbed by the fly-wheel as kinetic energy.

$$\therefore\ 46{,}200 = \frac{1}{2} \cdot \frac{3 \times 2{,}240}{32{\cdot}2} \times 25(N^2 - 100^2) \times \left(\frac{2\pi}{60}\right)^2,$$

whence $N = 107{\cdot}8$,

$\therefore$ change of speed $= 7{\cdot}8$ revs. per min.

EXERCISES. II

1. A cage weighing 1,000 lb. is being lowered down a mine by a cable. Find the tension in the cable (*a*) when the speed is increasing at the rate of 5 ft. per sec. per sec., (*b*) when the speed is uniform, (*c*) when the speed is diminishing at the rate of 5 ft. per sec. per sec. The weight of the cable itself may be neglected. [*Inst. C. E.*]

2. The mass of a balloon and its car is 4,000 lb.; it displaces 4,500 lb. of air. Find the acceleration with which the balloon will rise, and, assuming the mass of the car to be 1,000 lb., find the altered pull upon the suspending ropes. [*Inst. C. E.*]

3. A force of 15 lb. acts upon a mass weighing 250 lb. and increases its speed from 5 ft. per sec. to 40 ft. per sec. Calculate the time taken to effect this increase in speed and also the distance moved through in this time. Find the kinetic energy of the mass at the initial and final speeds, and show that the change in kinetic energy is equal to the work done by the accelerating force. [*Inst. C. E.*]

4. A bullet of mass $\frac{7}{11}$ of an ounce leaves a gun with a velocity of 1,200 ft. per sec. Find the average force expelling the bullet if the barrel of the gun is 42 in. long. [*Inst. C. E.*]

5. A circular drum has a weight of 3 tons, an outside diameter of 7 ft., and a radius of gyration of 3 ft. A rope 1,000 ft. long, and weighing 2·5 lb. per ft., hangs from the drum and carries a cage which weighs 2 tons at its full speed.

A constant torque T is made to turn the drum, and moves the cage a distance of *100 ft. in 20 sec. from rest. Find the value of the torque.* [*I. Mech. E.*]

6. A weight of 800 lb. falls from a height of 6 ft. on to a pile which weighs 600 lb. Assuming the pile is inelastic, find the velocity with which it begins to move, and also find the mean resistance of the earth to penetration if the pile is driven 0·8 in. into the ground by the blow. [*Inst. C. E.*]

7. A mass weighing 1 ton is pulled along a smooth horizontal plane with a constant force of 0·1 ton for a distance of ½ mile, and then up an incline of 1 in 50 for another ½ mile. Assuming the frictional and air resistance to be constant and equal to 50 lb., find the time taken to reach the top of the incline from the starting-point. [*I. Mech. E.*]

8. A fly-wheel weighs 2 tons and has a radius of gyration of 4 ft. It is fixed to the crankshaft of an engine which at 150 revs. per min. generates 20 horse-power. Assuming the mean torque on the crank to remain constant at all speeds, find the time for the engine to get up full speed, neglecting all masses but the fly-wheel. [*I. Mech. E.*]

9. On a circular railway there are twenty stations, and a train weighing 150 tons completes the circuit in fifty minutes. If the brakes are applied at each station when the train is travelling at 25 miles per hour, find the cost of stopping the train per hour, assuming the value of one horse-power-hour is 3*d*. [*I. Mech. E.*]

10. Calculate the kinetic energy stored in a fly-wheel which is making 180 revs. per min., it being assumed that the mass of the wheel, which is 2·6 tons, acts at a radius of 2 ft. 8 in. If the speed of the wheel falls to 15 revs. per min. in 55 sec., express in horse-power the mean rate at which the fly-wheel gives out energy.

11. A locomotive developing 1,000 horse-power pulls a train weighing 300 tons along a level track. Assuming that the frictional resistances for speeds above 30 miles per hour can be calculated from $R = 0{\cdot}5(V - 20)$, where R is the resistance in pounds per ton and V is the speed in miles per hour, find the maximum speed of the train.

12. A railway train weighing 600 tons is pulled up an incline 4½ miles long having a slope of 1 in 120. The frictional resistances may be assumed constant at 15 lb. per ton. If the engine exerts a constant pull and the speeds at the beginning and end of the incline are respectively 55 and 32 miles per hour, find the magnitude of the pull in tons and the time taken to climb the incline.

Express in horse-power hours the work done by the locomotive during the climb.

13. A locomotive pulls at constant speed of 40 miles per hour a train of 11 coaches, each weighing 30 tons. Find the pull of the locomotive and also its effective horse-power when running along the level against resistances of 13 lb. per ton.

If the train is relieved of the four back coaches by 'slipping' them, find: (*a*) the acceleration of the main part of the train; (*b*) the

retardation of the slipped coaches; (c) the distance travelled by the slipped coaches before coming to rest.

14. The shaft of a coal-mine is 800 ft. deep, and the cage in which the coal is raised weighs 7 cwt. and hangs at the end of a rope which weighs $2\frac{1}{2}$ lb. per ft. A truck filled with coal and thus weighing 15 cwt. is placed in the cage at the bottom of the shaft and is raised to the surface in 25 sec. Calculate the mean effective horse-power of the engine which works the winding shaft.

If the cage starts with an acceleration of 10 ft. per sec. per sec., what is the horse-power of the engine at the end of one second?

15. If, in Question 14, the acceleration is constant at 10 ft. per sec. per sec. during the period of acceleration, and the retardation when arriving at the top is that due to gravity, the rest of the motion being at uniform speed, find the times of acceleration, constant speed, and deceleration, and also the magnitude of the constant velocity.

Plot on a time-base curves showing: (a) the height through which the cage is raised; (b) the horse-power exerted by the engine.

16. Find the kinetic energy stored in the fly-wheel of a rolling-mill engine if the speed of rotation of the wheel is 100 revs. per min. and the mass of the wheel is 25 tons, at a radius of gyration of 9 ft.

If while a billet passes through the rollers the speed of the engine reduces from 100 to 85 revs. per min. in 5 sec., calculate the rate at which energy is withdrawn from the fly-wheel.

17. A fly-wheel having a radius of gyration of 10 in. is rotating at 852 revs. per min. A brake applied to the outside of the rim reduces the speed to 300 revs. per min. at a constant rate in 2 min. Find the brake resistance, if the wheel weighs 140 lb. and the rim diameter is 24 in. [*Lond. B.Sc.*]

18. Prove that when a mass of W lb. moves in a circle of radius r ft., with a velocity of v miles per hour, the force in lb. acting on the mass towards the centre of the circle is

$$P = \frac{2 \cdot 15 W v^2}{gr} \text{ lb.}$$

A turbine rotor 7 in. in diameter has fixed to its rim a small blade which weighs 0·06 lb. The wheel makes 25,000 revs. per min. Find the force holding the blade to the rim. [*I. Mech. E.*]

19. A fly-wheel weighs 48 tons and has a radius of gyration of 6 ft. It runs at 420 revs. per min. Find: (a) the stored energy of the fly-wheel in horse-power-second units; (b) the torque exerted on the shaft to stop the fly-wheel in 12 min. [*I. Mech. E.*]

20. A wagon of total weight 2,000 lb. has four wheels of 1·5 ft. radius each weighing 200 lb., and each wheel has a radius of gyration of 1·35 ft.

The wagon starts from rest at the top of an incline having a slope of 1 in 15. There is a resistance to motion, which may be assumed constant, of 11 lb. per ton. Determine the velocity of the wagon after it has moved a distance of 2,000 ft. [*I. Mech. E.*]

21. A crane lifts 1 ton from the ground by a chain, 160 ft. long, weighing 1 lb. per ft. run, which is wound initially on a barrel of radius 9 in. to the centre of the chain. The weight of the barrel, etc., which revolve with the axle, is 300 lb. and the radius of gyration is 15 in. Find the torque in lb.-ft. necessary to give a linear acceleration to the ton weight of 1 ft. per sec. per sec. Calculate the average horse-power to drive the gear and lift the weight in the first 3 sec. of the lift. [*Lond. B.Sc.*]

22. The radius of gyration of a fly-wheel weighing 10 tons is 6 ft. Calculate the energy the fly-wheel stores at 300 revs. per min.; the uniform torque which must act on the shaft to produce the speed of 300 revs. per min. by uniform angular acceleration in $1\frac{1}{2}$ min.; and the brake tension which must be applied to two diametral rods connecting two cast iron brake blocks to stop it in 3 min., assuming that the radius of the surface on which the blocks act is 6·0 ft. and that the coefficient of friction is 0·2. [*Lond. B.Sc.*]

23. Show that when a body is moving about a fixed axis with an angular acceleration α under the influence of a turning-moment T, then

$$T = I\alpha,$$

where I is the moment of inertia of the body with reference to the axis of rotation.

The rotor of a hydraulic turbine weighs 25 tons and has a radius of gyration of 5 ft. When running at 200 revs. per min. the rotor is suddenly relieved of part of its load so that its speed rises to 205 r.p.m. in 1 sec. Find the unbalanced turning-moment exerted, assuming this uniform throughout the change of speed. [*Lond. B.Sc.*]

24. A fly-wheel, the rim of which has a mean diameter of 6 ft. and a weight of $1\frac{1}{2}$ tons, is connected to a shaft of an engine which develops 20 horse-power when running at 150 revs. per min.

Assuming the work done per stroke of the engine is constant and that there is no work done by the crankshaft, or by friction, determine the time required for the engine, starting from rest, to reach its speed of 150 revs. per min.

If the frictional resistance of the engine remains constant and the mechanical efficiency at 20 horse-power is 80 per cent., find the time taken to stop the engine after the steam is shut off. [*Lond. B.Sc.*]

25. A cage, which weighs $2\frac{1}{2}$ tons, is hung from a steel rope, the upper end of which is wound upon a horizontal drum 54 in. in diameter. The radius of gyration of the drum is 25 in. The drum is rotated by an electric motor, which applies a steady torque of 15,000 ft.-lb. to its shaft. Assuming that 5 per cent. of this torque is wasted in overcoming the friction of the bearings, etc., and that the cage is hanging free at the instant that the drum begins to revolve, find: (*a*) the acceleration of the cage; (*b*) the time needed to raise the cage 120 ft.; (*c*) the tension in the rope. The weight of the drum may be assumed to be 1 ton. [*Lond. B.Sc.*]

26. A locomotive pulls a train weighing 450 tons up a gradient of 1 in 80 at 30 miles per hour. Frictional resistances above 20 miles per hour are $0 \cdot 2v$ lb. per ton where v is the speed in feet per second. When the train reaches the top of the gradient, find the time that elapses before the speed reaches 60 miles per hour.

27. A locomotive and train together weigh 350 tons; find the horse-power of the locomotive when the speed on the level is constant at 60 m.p.h. and frictional resistances are 20 lb. per ton.

If the train comes to a bank 3 miles long rising 1 in 135, find the time taken to reach the top of the bank and the speed at the top—

(1) if the pull remains constant,

(2) if the horse-power is constant.

CHAPTER III

MECHANISMS AND VELOCITY DIAGRAMS

§ 35. **Mechanisms.** A machine, as already explained, may be regarded as an agent for transmitting or modifying energy. In order to design a machine it is necessary to consider the exact nature of the work the machine has to perform, and, assuming this function to be known, a skeleton or outline of the machine may be drafted from which the motions between the respective parts may be considered. When this scheme has been drafted, the parts require to be modified in shape and size so as to transmit the forces involved. This part of the subject is adequately dealt with in books on machine design. The object of the present volume is to study chiefly the motions that take place between the different parts of machines and where necessary to include the static and kinetic forces involved.

A mechanism may be regarded as a skeleton machine which is not required to transmit energy, but merely to reproduce exactly the motions that take place in an actual machine. The various parts of a mechanism may be termed links or elements. The representation of a mechanism is conveniently accomplished by means of a skeleton or line diagram, this diagram being known as a configuration diagram. For convenience, configuration diagrams will be used to illustrate the mechanisms about to be discussed.

§ 36. **Motion of a Link.** Before describing the construction of vector (in this case velocity) diagrams, the motion of a link about one extremity should be fully understood.

In Fig. 14, let AB represent a link capable of rotating about the end A. Let the direction of rotation be clockwise, as shown by the arrow at B. For the configuration shown, B is moving instantaneously perpendicular to AB, and the motion of B relative to A may be represented by the vector ab. This is perhaps more easily visualized when it is remembered that B cannot approach or recede from A, and consequently the only possible motion of B relative to A is

in a direction perpendicular to AB. The statement that B cannot approach nearer to A is made on the assumption that AB is rigid, i.e. rigid in the sense that the length of the link does not appreciably alter.

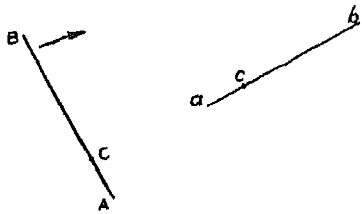


Fig. 14.

Let ω = angular velocity of the link AB about A.

Then the velocity of B relative to $A = \omega . AB$.

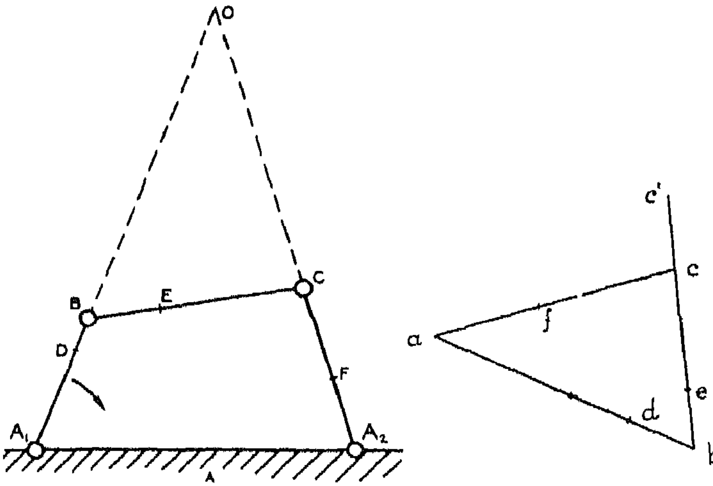
Similarly, the velocity of an intermediate point such as $C = \omega . AC$. The velocity of C relative to A is represented by the vector ac and it is easily seen that $\frac{ac}{ab} = \frac{AC}{AB}$. Thus the point c on the vector ab divides it in the same ratio that C divides the link AB.

The velocity of A relative to B is represented by ba, although A may be a fixed point. The motion between A and B is only relative and it is immaterial whether the link moves about A in a clockwise direction or about B in a clockwise direction.

§ **37. Quadric Cycle Chain.** Let Fig. 15 (*a*) represent a four-bar or quadric cycle chain. This mechanism consists of a fixed link A_1A_2, movable links A_1B and A_2C, and a coupler or connecting link BC. The base or fixed part of the mechanism may be represented by A, and particular points on the fixed link by subscripts to A, as A_1 and A_2.

In practice the mechanism does not usually take the form as shown in Fig. 15 (*a*), but this serves as a very convenient example to explain the construction of the velocity diagram; moreover, by modifying the lengths of the links, various mechanisms commonly met with in practice may be recognized. Thus, if A_1B and A_2C are made equal, and small in comparison to A_1A_2, and if BC is made equal to A_1A_2,

the mechanism is that of the two coupled wheels of a locomotive.



(a) Fig. 15. (b)

§ 38. **Velocity Diagram for Quadric Cycle Chain.** In Fig. 15 (a), let $A_1 B$ be the driving link rotating at an angular speed ω radians per sec. Then, to find the velocity of C relative to A the fundamental velocity equation may be formed, thus:

velocity of C relative to A = velocity of C relative to B
+velocity of B relative to A,

or $$ac = bc + ab = ab + bc.$$

The velocity of B relative to A_1 (or A) $= \omega . A_1 B$.

In Fig. 15 (b), ab is drawn perpendicular to $A_1 B$ and to represent $\omega . A_1 B$ to scale. The vector ab represents the velocity of B relative to A; to this must be added a vector bc to represent the velocity of C relative to B. The magnitude of this velocity is unknown, but the direction is known since C, relative to B, must move perpendicular to BC; hence bc' is drawn perpendicular to BC to represent the velocity of C relative to B; the vector bc is as yet undetermined, but the direction bc' is known, that is, the point c on the vector bc' is not yet located.

The sum of the two vectors ab and bc (when known) is ac, as is seen from the fundamental velocity equation. Hence from a a vector ac is drawn perpendicular to A_2C to represent the velocity of C relative to A_2. The vector ac meets bc' in c, thus locating the point c. The complete velocity diagram is the triangle abc. The velocity of intermediate points in the links such as D, E, and F is readily obtained by dividing the vectors in the same ratio as the points divide the links. Thus

$$\frac{A_1D}{A_1B}=\frac{ad}{ab}; \qquad \frac{BE}{BC}=\frac{be}{bc}; \qquad \frac{A_2F}{A_2C}=\frac{af}{ac}.$$

§ 39. Angular Velocity of Links. When the velocity of one end of a link relative to the other end is known, the angular velocity of the link about either extremity is easily deduced. Thus the velocity of C relative to B is represented by bc, hence C relative to B is moving in a direction from b to c, or about B, C is moving in counter-clockwise direction.

Let ω_1 = angular velocity of link BC about B.

Then $\omega_1 . BC$ = velocity of C relative to $B = bc$.

$$\therefore \omega_1 = \frac{bc}{BC}.$$

The angular velocity of the link CB about C is in the direction from c to b, that is, in a counter-clockwise direction, and is of magnitude $\frac{cb}{CB}$, which is equal to ω_1. Hence the angular velocity of a link about one extremity is the same in magnitude and direction as the angular velocity about the other extremity. Since $bc = -cb$, the linear velocity of one extremity of a link about the other is equal in magnitude, but opposite in direction, to the linear velocity of the other extremity about the one.

The vector ac represents the velocity of C relative to A, hence C is moving in a direction from a to c, that is, in a clockwise direction about A_2.

Let ω_2 = angular velocity of link A_2C about A_2.

Then $\omega_2 . A_2C$ = velocity of C relative to $A_2 = ac$,

$$\therefore \omega_2 = \frac{ac}{A_2C}.$$

§ **40. Velocity of Rubbing at Pins.** The velocity of rubbing at a pin connecting two links depends upon the relative angular velocity of the links. The pin at A_1, Fig. 15 (a), connects the links $A_1 A_2$ and $A_1 B$, and since $A_1 A_2$ is fixed, the velocity of rubbing is the radius of the pin multiplied by the angular velocity of the link $A_1 B$.

Let r_1 = radius of pin at A_1.

Velocity of rubbing $= r_1 . \omega$.

Similarly, the velocity of rubbing of the pin at A_2 is $r_2 . \omega_2$, where r_2 is the radius of the pin at A_2 and ω_2 is the angular velocity of the link $A_2 C = \frac{ac}{A_2 C}$.

The velocity of rubbing at the pin B depends upon the relative angular velocity between the links BA_1 and BC. The velocity of A_1 relative to B is represented by ba, hence A_1, relative to B, may be considered as moving in a direction from b to a, that is, A_1 is moving in a clockwise direction about B. The velocity of C relative to B is represented by bc, hence C, relative to B, is moving in a direction from b to c, or C is moving in a counter-clockwise direction about B. Since the links BA_1 and BC are moving in opposite directions about B, the relative angular velocity between the links is the sum of the individual angular velocities.

Let r_3 = radius of pin at B;

$$\omega_1 = \frac{bc}{BC} = \text{angular velocity of link } BC \text{ about } B.$$

Then velocity of rubbing $= r_3(\omega + \omega_1)$.

Similarly at C, the link CB is rotating in a counter-clockwise direction and the link CA_2 in a clockwise direction. Since these links are moving in opposite directions about C, the relative angular velocity is the sum of the individual angular velocities.

Let r_4 = radius of pin at C.

Velocity of rubbing $= r_4(\omega_1 + \omega_2)$.

In the case of a pin joining two links moving in the same direction with different angular velocities, the velocity of

rubbing is the radius of the pin multiplied by the difference of the angular velocities.

§ 41. **Instantaneous Centre of Rotation.** Referring to the quadric cycle chain, Fig. 15 (*a*), for the configuration shown, B is moving instantaneously perpendicular to $A_1 B$, and, for a very small movement of B, the centre of rotation of B may be considered at any point in $A_1 B$ or $A_1 B$ produced. In a similar manner, C moving instantaneously perpendicular to $A_2 C$, may have its centre of rotation at any point in $A_2 C$ or $A_2 C$ produced. Let $A_1 B$ and $A_2 C$ be produced to meet at O, then O is known as the instantaneous centre of rotation for the link BC, and for the instant the whole link BC may be considered to be rotating about O.

Let ω = angular velocity of $A_1 B$ about A_1,
ω_2 = angular velocity of $A_2 C$ about A_2,
Ω = angular velocity of link BC about O.

Then velocity of $B = \omega . A_1 B = \Omega . OB$

$$\therefore \Omega = \omega . \frac{A_1 B}{OB}.$$

$$\text{Velocity of } C = \Omega . OC = \omega . \frac{A_1 B}{OB} . OC = \omega_2 . A_2 C,$$

$$\therefore \frac{OC}{OB} = \frac{\omega_2 . A_2 C}{\omega . A_1 B} = \frac{ac}{ab}.$$

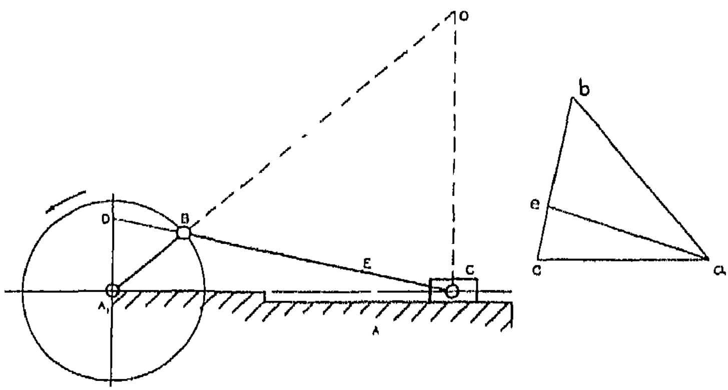
The triangles OBC and abc are similar, ab being perpendicular to OB (or $A_1 B$), bc perpendicular to BC, and ac perpendicular to OC; hence from the geometry of the two triangles $\frac{OC}{OB} = \frac{ac}{ab}$. Thus, the velocity of C or the angular velocity of $A_2 C$ may be found, either by constructing the velocity diagram or by the method of finding the instantaneous centre of rotation. The triangle OBC is, indeed, a velocity diagram.

The instantaneous centre method is perhaps easier to understand and to apply in certain cases, but apparently fails when $A_1 B$ and $A_2 C$ are parallel. Even when $A_1 B$ and $A_2 C$ are not parallel, but nearly so, the point O is liable to be at an inconvenient distance from BC. The advantages accruing

from the use of the velocity diagram are that it may be used for any configuration and that similar methods apply to the construction of acceleration diagrams, as will be seen in Chapter IV.

§ 42. **Slider Crank Chain.** The slider crank chain is a modification of the quadric cycle chain, by making one of the links of infinite length.

Referring to Fig. 15 (*a*), C moves in a circular arc with A_2 as centre; if the link A_2C is made of infinite length, C will move in a straight path perpendicular to A_2C. By reducing the length of the link A_1B, so that it may make complete revolutions, thereby becoming a crank, and by increasing the length of the coupler BC, a mechanism is obtained which will be readily recognized as that of the direct acting steam engine.



(*a*) Fig. 16. (*b*)

In Fig. 16 (*a*), let A_1B represent a crank rotating at constant angular speed in the direction of the arrow. Let C represent the crosshead which is rigidly attached to the piston (not shown) by a piston rod (also not shown). The crosshead, piston, and piston rod being rigidly fixed together and moving as one part of the mechanism, it is only necessary to consider the motion of the crosshead.

§ 43. **Velocity of Crosshead.** The velocity of the crosshead C, Fig. 16 (*a*), may be found by constructing the velocity diagram.

Velocity of C relative to A = velocity of C relative to B + velocity of B relative to A

or $$ac = bc+ab = ab+bc.$$

The vector ab is drawn perpendicular to A_1B and of magnitude proportional to $\omega . A_1B$, to represent the velocity of B relative to A, where ω is the angular velocity of the crank A_1B. The velocity of C relative to B is unknown in magnitude, but in direction is perpendicular to BC; hence bc is drawn perpendicular to BC. The velocity of C relative to A is unknown in magnitude, but in direction C is constrained by the guides to move in a straight path, hence, drawing ac parallel to the direction of motion of C, ac meets the vector bc at c. The velocity diagram is abc and the velocity of the crosshead is represented by ac.

Using the method of instantaneous centre of rotation, B can be considered as rotating about any point in A_1B or A_1B produced and C as rotating about a point on a line perpendicular to the guides and passing through C. Let A_1B produced and the perpendicular at C meet at O. The triangles OBC and abc are again similar and OBC is thus a velocity diagram.

Let Ω = velocity of link BC about O.

Then velocity of $B = \Omega . OB = \omega . A_1B$.

$$\therefore \Omega = \omega . \frac{A_1B}{OB}.$$

$$\text{Velocity of } C = \Omega . OC = \omega . A_1B . \frac{OC}{OB}.$$

The velocity of the crosshead may be obtained in a very convenient manner by producing CB to meet a perpendicular through A_1 in D. The triangles OBC and A_1BD are similar, hence

$$\frac{OC}{OB} = \frac{A_1D}{A_1B}$$

and the velocity of C may be written

$$\omega . A_1B . \frac{A_1D}{A_1B} \quad \text{or} \quad \omega . A_1D.$$

Velocity of crosshead $= \omega . A_1D$.

$A_1 D$ is measured to the same scale as that in which $A_1 B$ represents the crank radius.

The angular velocity of the connecting-rod is readily obtained from the velocity diagram and in magnitude is $\frac{bc}{BC}$.

Call this angular velocity ω_1.

Let r_1 = radius of crank-shaft at A_1,

r_2 = radius of gudgeon pin at C,

r_3 = radius of crank pin at B.

Then velocity of rubbing at $A_1 = \omega r_1$,

velocity of rubbing at $C = \omega_1 r_2$,

velocity of rubbing at $B = (\omega + \omega_1) r_3$.

The velocity of a point E on the connecting-rod is found by dividing bc at a point e such that $\frac{BE}{BC} = \frac{be}{bc}$. The velocity of E, relative to A, in magnitude, is represented by ae, and in direction is from a to e.

EXAMPLE 1. Find the velocity of the slider E in the mechanism shown in Fig. 17 (a) when the crank $O_1 A$ rotates at 1 rev. per sec. [*Lond. B.Sc.*]

Let Fig. 17 (a) represent the configuration diagram. Velocity of A relative to $O = 2\pi \times 2\frac{1}{2} = 15{\cdot}7$ in. per sec. The velocity of E cannot be directly determined from that of A, but must be found from the velocity of D, which in turn is found from that of B.

Velocity of B relative to O = velocity of B relative to A + velocity of A relative to O;

or $$ob = ab + oa = oa + ab.$$

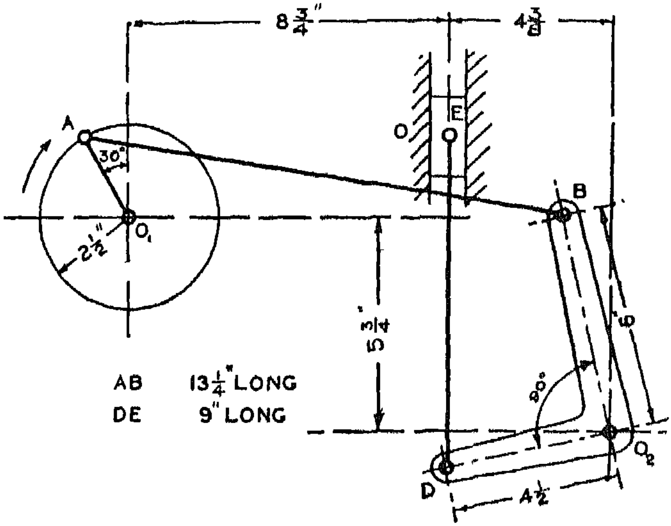
In Fig. 17 (b), oa is drawn perpendicular to $O_1 A$ and of length proportional to the velocity of A relative to O, i.e. 15·7 in. per sec. The velocity of B relative to A is perpendicular to AB; hence ab is drawn perpendicular to AB. The velocity of B relative to O is perpendicular to $O_2 B$ and ob is drawn perpendicular to $O_2 B$. The velocity of B relative to O is represented by ob, which scales 13·1 in. per sec.

The points B and D rotating about O_2 have velocities proportional to their distances from O_2, and the velocity of D relative to $O_2 = \frac{4\frac{1}{2}}{6} \times 13{\cdot}1 = 9{\cdot}83$ in. per sec.

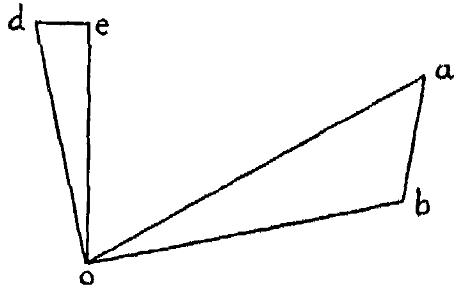
The vector od is drawn perpendicular to $O_2 D$ (or to ob), its length representing 9·83 in. per sec.

Velocity of E relative to O = velocity of E relative to D
+velocity of D relative to O,

or $$oe = de + od = od + de.$$



(a)

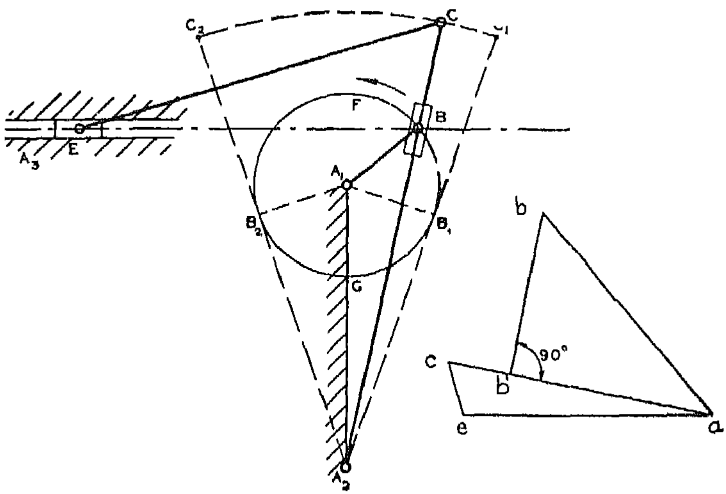


(b)

Fig. 17.

The velocity of D relative to E is perpendicular to DE and de is drawn perpendicular to this link. The velocity of E relative to O is in the direction of the guides constraining the motion of E, hence oe is drawn in this direction. The vector oe represents the velocity of E relative to O and scales 9·6 in. per sec.

§ 44. Crank and Slotted Lever Mechanism. The mechanism shown in Fig. 18 (*a*) represents one form of quick return motion used for shaping, slotting machines, etc. It consists of a rotating crank $A_1 B$, rotating round a fixed centre A_1, an oscillating lever $A_2 C$, which is caused to oscillate about A_2 by the sliding of the block at B as $A_1 B$ rotates; a link EC attached to C and to E, which corresponds to the ram of the machine, causes the ram to have a reciprocating motion. The cutting tool, which is attached to the ram,



(*a*) Fig. 18. (*b*)

will be at the ends of the stroke when the oscillating lever $A_2 C$ occupies the extreme positions $A_2 C_1$ and $A_2 C_2$. The corresponding positions of the crank are $A_1 B_1$ and $A_1 B_2$. As the crank moves from the position $A_1 B_1$ through B and F to $A_1 B_2$, the ram makes a complete stroke (cutting stroke), and the crank, continuing its motion from the position $A_1 B_2$ through G to $A_1 B_1$, causes the ram to make a complete stroke in the opposite direction; this is the return stroke. The crank, rotating at constant speed, moves from the position $A_1 B_2$ through G to $A_1 B_1$ in less time than it moves from $A_1 B_1$

through F to $A_1 B_2$; hence the return stroke is made in less time than the cutting stroke.

$$\frac{\text{time of cutting}}{\text{time of return}} = \frac{\text{arc } B_1 F B_2}{\text{arc } B_2 G B_1}.$$

§ **45. Velocity of Cutting.** For the given configuration in Fig. 18 (*a*), the velocity of E is found by constructing the velocity diagram. Assuming the direction of rotation as shown by the arrow, the velocity of B relative to A can be represented by the vector ab, Fig. 18 (*b*). The velocity of B can be split up into two component velocities, one along $A_2 C$ and the other perpendicular to this link. These components are shown as $b'b$ and ab', which are mutually perpendicular, $b'b$ being parallel to $A_2 C$ and ab' perpendicular to $A_2 C$. The vector $b'b$ represents the velocity of sliding of the block B along the oscillating lever $A_2 C$.

The velocity of C is readily found when the velocity of B perpendicular to $A_2 C$ (represented by ab') is known, by making

$$\frac{ab'}{ac} = \frac{AB}{AC}.$$

Velocity of E relative to A = velocity of E relative to C
+ velocity of C relative to A,

or $$ae = ce + ac = ac + ce.$$

The velocity of E relative to C is perpendicular to EC and this is represented by the vector ce drawn perpendicular to CE. The velocity of E relative to A is in the direction of the guide for the ram (in this case horizontal) and is represented by ae drawn horizontal. The velocity of the ram at E is thus proportional to the vector ae.

EXAMPLE 2. In the mechanism shown in Fig. 19 (*a*), A_1 and A_2 are fixed centres. The crank $A_2 C$ rotates at a uniform speed of 12 radians per second, the end C being pivoted to a block which can slide along $A_1 D$. The block E is constrained to move along a path represented by EA_1 by means of the link DE; $A_1 D$ rotates about the fixed centre A_1. For the configuration shown, find the velocity of E. $A_1 A_2 = 2\frac{1}{2}$, $A_2 C = 3\frac{3}{4}$, $A_1 D = 8$, and $DE = 16$ in.

Velocity of C relative to $A = 12 \times 3\frac{3}{4} = 45$ in. per sec.

In Fig. 19 (*b*), ac is drawn to represent the velocity of C relative

to A and is proportional to 45 in. per sec. This velocity has two components, along and perpendicular to A_1D. These components are found by drawing ac' and $c'c$ mutually perpendicular, at right angles and parallel to A_1D. The vector $c'c$ is the velocity of sliding of the block C on A_1D.

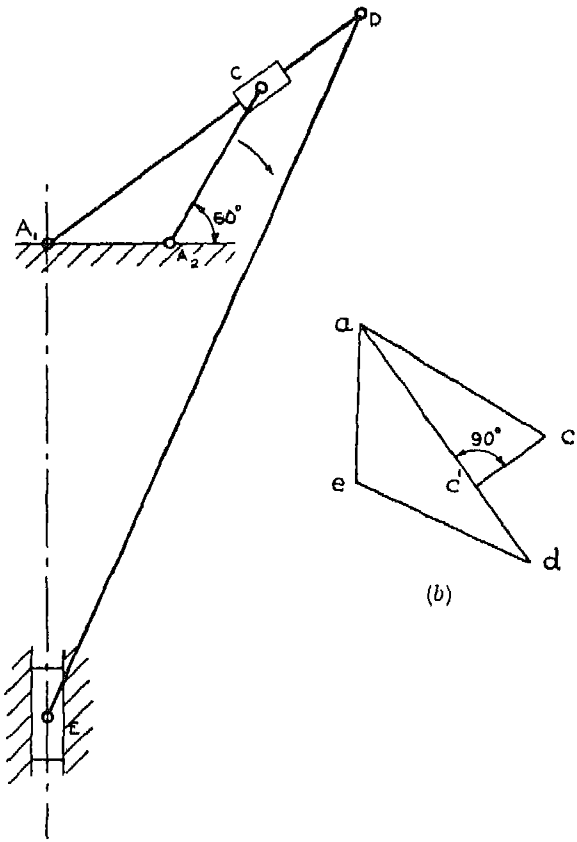


Fig 19.

The velocity of D is found by making $\frac{ac'}{ad} = \frac{A_1C}{A_1D}$.

$$ac' = 41{\cdot}3 \text{ in. per sec., and } \frac{A_1C}{A_1D} = \frac{5{\cdot}44}{8};$$

$$\therefore\ ad = \frac{8}{5{\cdot}44} \times 41{\cdot}3 = 60{\cdot}8 \text{ in. per sec.}$$

Velocity of E relative to A = velocity of E relative to D
+velocity of D relative to A,

or $$ae = de + ad = ad + de.$$

The velocity of E relative to D is perpendicular to ED, hence de is drawn perpendicular to this link. E is constrained to move in a direction EA_1 and the velocity of E relative to A is represented by a vector ae drawn parallel to EA_1. The vector ae scales 33·5 in. per sec. or 2·79 ft. per sec.

An alternative method of finding the point b' Fig. 18 (b) on the velocity diagram is to imagine a point B' on A_2C immediately under B on the crank A_1B. In Fig. 18 (a) the point B can represent two points:

(1) a point B at the end of the crank A_1B;
(2) a point B' on the swinging link A_2C but immediately beneath B.

To find the velocity of B' the velocity equation can be formed thus:

Velocity of B' relative to A = velocity of B' relative to B
+velocity of B relative to A.

The velocity of B relative to A is represented by ab perpendicular to A_1B and in magnitude is proportional to $\omega . A_1B$, where ω is the angular velocity of A_1B. The velocity of B' relative to B is along A_2C and hence bb' is drawn through b and parallel to A_2C. The point b' is not yet located. The velocity of B' relative to A is perpendicular to A_2C, since B' is a point on A_2C. Hence ab' is drawn perpendicular to A_2C through a and the intersection of ab' and bb' locates the point b'. The remainder of the velocity diagram is drawn in the manner already described.

§ **46. Whitworth Quick-Return Motion.** In the mechanism of the Whitworth quick-return motion shown in Fig. 20, a rotating link A_2B rotates at constant speed about the fixed centre A_2. The free end of the link is attached to a block at B, which can slide in a slot in the rotating link BA_1C; this link rotates about the fixed centre A_1. Attached to C is a link CD (shown incomplete) which is attached to the ram of the machine. The cutting tool is rigidly fixed to the ram,

hence motion is imparted to the tool by means of the link CD. The line of stroke of the ram is along $B_1 B_2$ produced. When the link $A_2 B$ occupies the position $A_2 B_1$, C is in its extreme position on the right and the ram is at one end of its stroke; similarly when B has moved to B_2, C is in its extreme position on the left and the ram is at the other end of its stroke.

Assuming the direction of rotation of $A_2 B$ to be clockwise,

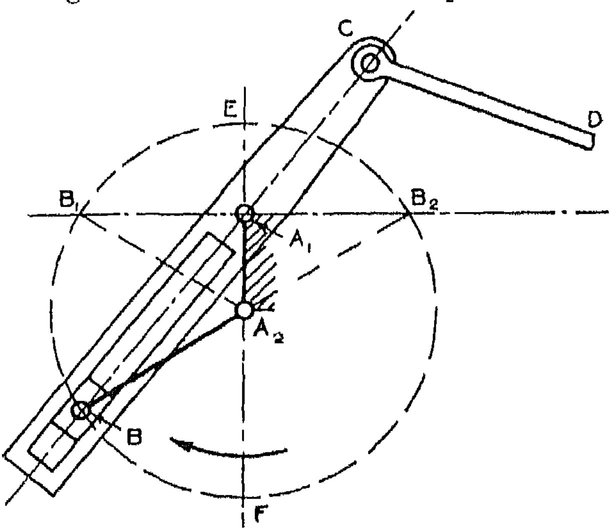


Fig. 20.

as shown by the arrow, the cutting stroke occurs while the ram is moving to the right, i.e. while the link $A_2 B$ moves from the position $A_2 B_2$ to $A_2 B_1$ through the point F. The return stroke occurs while B moves from B_1 to B_2 through the point E.

$$\therefore \frac{\text{time of cutting}}{\text{time of return}} = \frac{\text{arc } B_2 F B_1}{\text{arc } B_1 E B_2}.$$

The maximum velocity of cutting occurs approximately when B is at F, and the maximum velocity of return approximately when B is at E.

In modern machines the link $A_2 B$ is replaced by a spur wheel which carries a pin and block B. The spur wheel is driven by a pinion which is driven either direct from the cone pulleys or through gearing.

The construction of the velocity diagram for the Whitworth quick-return motion does not present any unusual difficulty and the student should have no difficulty in constructing one.

§47. Ratio of Speeds of Quick-Return Motions. Referring to the crank and slotted lever quick-return motion, Fig. 18 (*a*), the length of the stroke of the cutting tool at E can be varied by altering the effective length of the crank A_1B. This crank frequently consists of a radial slot in a wheel, and the position of the point B can be varied, thus altering the length of the crank. When the stroke is small, the crank A_1B is small and the arcs B_1FB_2 and B_2GB_1 become more nearly equal. Consequently the ratio of time of cutting to time of return varies with the stroke and is a maximum when the stroke is a maximum and a minimum when the stroke is a minimum.

In the case of the Whitworth quick-return, Fig. 20, the length of the stroke is varied by altering the point of attachment of the coupler CD; thus the distance A_1C is variable depending upon the stroke. The ratio of the time of cutting to time of return remains constant for all strokes since the arcs B_2FB_1 and B_1EB_2 remain unaltered due to alteration of stroke. This enables the Whitworth machine to be run at a higher speed than the crank and slotted lever, in general, for the same speed of cutting. To compare the speeds of the two machines:

Let T_1 = time of cutting in seconds,
t_1 = time of return in seconds,
N_1 = speed in revolutions per second
for the Whitworth machine.

Let T_2 = time of cutting in seconds,
t_2 = time of return in seconds,
N_2 = speed in revolutions per second
for the crank and slotted lever machine.

Let s = stroke for both machines.

Average cutting speed $= \dfrac{\text{stroke}}{\text{time of cutting}}$, and this is the same for both machines.

Hence $$\frac{s}{T_1} = \frac{s}{T_2} \quad \text{or} \quad T_1 = T_2.$$

Now T_1+t_1 is the time for cutting and return strokes in the Whitworth machine and is equal to the time for one revolution of the crank.

Hence $\dfrac{}{N_1} \quad T_1+t_1$ and similarly, $\dfrac{}{N_2} \quad T_2+t_2$.

$$\therefore \frac{N_1}{N_2} = \frac{T_2+t_2}{T_1+t_1} = \frac{T_2\left(1+\dfrac{t_2}{T_2}\right)}{T_1\left(1+\dfrac{t_1}{T_1}\right)} = \frac{1+\dfrac{t_2}{T_2}}{1+\dfrac{t_1}{T_1}}.$$

Assuming the ratio of $\dfrac{T_1}{t_1}$ is constant and equal to 2 for the Whitworth machine and that the ratio $\dfrac{T_2}{t_2}$ for the crank and slotted lever can be varied between 1 and 2, values of $\dfrac{N_1}{N_2}$ can be calculated for different values of $\dfrac{T_2}{t_2}$ as in the following table:

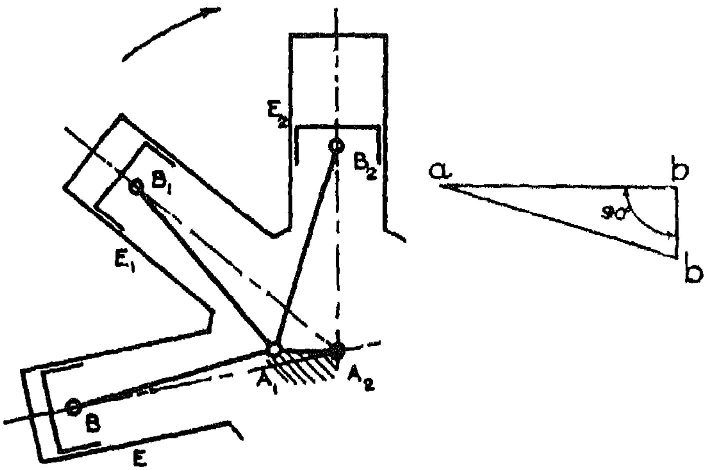
$\dfrac{T_2}{t_2}$	$\dfrac{N_1}{N_2} \quad \dfrac{1+\dfrac{t_2}{T_2}}{1+\dfrac{t_1}{T_1}}$
2·0	1·0
1·8	1·038
1·6	1·083
1·4	1·142
1·2	1·222
1·0	1·333

From the above table it will be seen that the Whitworth machine can, in general, be run at a higher speed than the crank and slotted lever machine for the same stroke and average cutting speed.

§ 48. **Rotary Aero-Engine.** This mechanism is derived from the slider crank chain and is termed an inversion of the slider crank. The rotary engine usually consists of seven or nine cylinders all revolving in the same plane or in parallel planes. These revolving cylinders form a balanced system.

In Fig. 21 (*a*) three cylinders E, E_1, and E_2 are shown which rotate about the fixed centre A_2. The crank of the engine is common to all cylinders and is fixed; it is represented by $A_1 A_2$. The pistons B, B_1, and B_2 have a reciprocating motion relative to their respective cylinders E, E_1, and E_2, since each piston is attached, by its connecting-rod, to the crank at A_1.

The velocity diagram shown in Fig. 21 (*b*) is constructed for cylinder E_2. Let ω = angular velocity of engine. Then velocity of B_2 relative to A_1 is perpendicular to $A_1 B_2$. Let



(*a*) Fig. 21. (*b*)

this be represented by the vector ab, drawn perpendicular to $A_1 B_2$. But the piston at B_2 has two component velocities, one in a direction from B_2 to A_2 and the other perpendicular to this direction; this latter component is due to the piston rotating with the cylinder E_2 which in turn rotates about A_2.

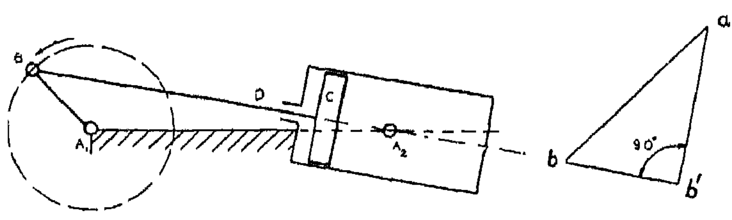
The vectors $b'b$ and ab' drawn mutually perpendicular represent these component velocities. The velocity of B_2 in a direction perpendicular to $A_2 B_2$ is $\omega . A_2 B_2$, hence ab' represents to scale $\omega . A_2 B_2$; the absolute velocity of the piston is represented by ab, and the velocity of sliding of the piston in the cylinder by $b'b$.

§ **49. Inversions of Mechanism.** The slider crank chain shown in Fig. 16 (*a*) can be converted into two other mechan-

isms by fixing in turn the crank A_1B and the connecting-rod BC. Mechanisms obtained in this way are termed inversions of the slider crank.

It will be noticed that sliding occurs between the block at C, on the end of the connecting-rod remote from the crank, and the frame. An example of an inversion in which the crank is fixed has just been explained in § 48, i.e. the rotary aero-engine. Further examples of inversions in which the crank is fixed are the crank and slotted lever mechanism and the Whitworth quick-return motion. Referring to the crank and slotted lever mechanism in Fig. 18 (*a*), it is seen that sliding occurs between the block B at the end of the link A_1B, which corresponds to the connecting-rod of the slider crank chain modified in length, and the oscillating link A_2C which may be regarded as corresponding to the frame of the slider crank chain. In the Whitworth quick-return motion, Fig. 20, it is readily seen that A_1A_2, A_2B, and BA_1C correspond to the crank, connecting-rod, and frame respectively of the slider crank chain.

An example of an inversion in which the connecting-rod is fixed is that of the oscillating or trunnion engine, shown in



(*a*) Fig. 22. (*b*)

Fig. 22 (*a*). The comparison between this and the slider crank chain is fairly obvious. The cylinder of the trunnion engine oscillates about the trunnion axis A_2. The piston C reciprocates in the cylinder and is attached to the crank A_1B by the piston rod BD.

The velocity of sliding of the piston in the cylinder and the

angular velocity of the cylinder about the trunnion axis is readily found from the velocity diagram.

Let ω = constant angular velocity of crank.

Velocity of B relative to $A_1 = \omega . A_1 B$.

This is represented by the vector ab. The crank pin B has two component velocities, one in the direction $A_2 B$ and the other perpendicular to this direction. These components are represented by $b'b$ and ab' respectively, which are mutually perpendicular.

The velocity of sliding of the piston is represented by $b'b$, and the angular velocity of the cylinder is $\dfrac{ab'}{A_2 B}$.

§ **50. Principle of Work.** Neglecting friction, the work done by an effort applied to a machine is equal to the work done by the machine; the work done by the machine may be regarded as the work done in overcoming a resistance. Consider Fig. 15 (a), and imagine a force P applied at B perpendicular to $A_1 B$; the work done by $P = P \times$ distance moved by B. Hence, the work done per second $= P \times$ velocity of B in inches or feet per second.

The force P at B is capable of overcoming a resistance R at C, applied perpendicular to $A_2 C$. The work done by the machine per second $= R \times$ velocity of C. Assuming no loss due to friction and that the inertia forces may be neglected,

$$P \times \text{velocity of } B = R \times \text{velocity of } C.$$

These velocities are readily obtained from the velocity diagram.

The forces P and R have been assumed perpendicular to their respective links; if, however, these forces are not perpendicular to their respective links, the components of the forces perpendicular to the links are only effective, and these components must only be used.

As a further example, let P be the effective force at the crosshead of the engine mechanism in Fig. 16 (a). Let R = resistance at B, perpendicular to $A_1 B$.

Then $\quad P \times \text{velocity of } C = R \times \text{velocity of } B.$

EXAMPLE 3. In Fig. 17 (*a*) a torque of 2,000 lb.-in. is driving the shaft at O_1. Find the resistance overcome at E.

$$\text{Effective force at } A = \frac{2{,}000}{2\frac{1}{2}} = 800 \text{ lb.}$$

Let R = resistance at E in lb.

$$\text{Then} \qquad 800 \times \text{velocity of } A = R \times \text{velocity of } E.$$

$$800 \times 15{\cdot}7 = R \times 9{\cdot}6,$$

$$\therefore \; R = 1{,}310 \text{ lb.}$$

EXERCISES. III

1. A reciprocating engine has a crank AB, $1\frac{1}{4}$ ft. long, and a connecting-rod BC, 5 ft. long. The crank revolves at 200 revs. per min. Find the velocity of the piston when the crank has turned through an angle of 60° from the inner dead centre.

Find also the velocity of a point D on the connecting-rod when CD is 1 ft.

2. The diameters of the crank-shaft, crank pin, and crosshead pin of the engine in Question 1 are 5, $5\frac{1}{2}$, and $3\frac{1}{2}$ in. respectively. Find the velocities of rubbing at each of these pins in ft. per min.

3. In an ordinary steam-engine mechanism the stroke of the piston is one-half the length of the connecting-rod. Assuming the crank-shaft to turn uniformly, draw a diagram to give the velocity of the piston at any instant. [*I. Mech. E.*]

4. In a four-bar mechanism the lengths of the links are as follows: *a*, 2 in.; *b*, 4 in.; *c*, 4·5 in.; *d*, 6 in., the latter being fixed. Find the velocity of a point on *c* at one inch from the pin connecting the links *c* and *d*, when the pin connecting the links *a* and *b* is moving at 6 ft. per sec. and the link *a* is at right angles to *d*.

5. A mechanism consists of four bars connected by pin joints. The lengths of the bars are: AB, 1 ft.; BC, 6 ft.; CD, 3 ft.; DA, 7 ft. AB and CD are both on the same side of DA, which is fixed, and AB has a speed of 20 revs. per min.

For the configuration in which the angle BAD is 90°, find the angular velocity of CD.

Find also the position and linear velocity of a point E on BC which has least velocity relative to DA.

6. A crank OP revolves with uniform speed of 120 revs. per min. about the fixed point O and drives, by means of the pin P, a slotted crank APB, which revolves about the fixed point A. OP is 12 in., AB is 18 in., and the distance OA is such that the minimum speed of B is half the maximum speed. Find the distance OA, and determine the velocity of the point B when the crank is at right angles to OA. [*I. Mech. E.*]

7. In the mechanism shown in Fig. 23, the link AB can swing about the fixed point A. The point C can travel along the axis CE and the point D along the axis AD, which is at right angles to CE. $AB:BD:CB:AE$ as $1:2:1{\cdot}5:0{\cdot}75$.

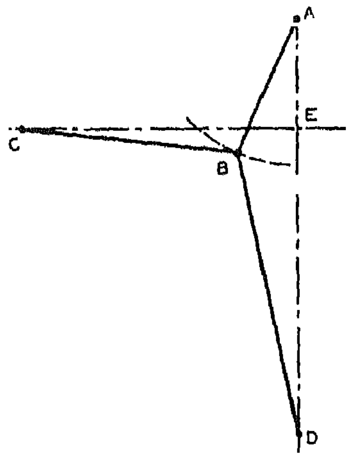


Fig. 23.

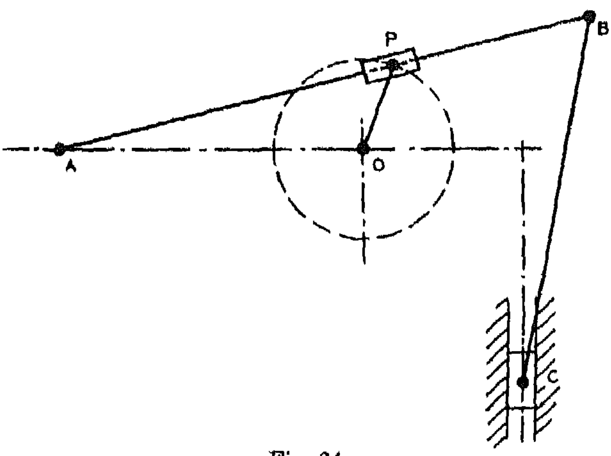


Fig. 24.

Determine the velocity-ratio of D to C when the angle DAB is 20°. Also find the position of the link AB for which the velocity-ratio of D to C is unity. [*I. Mech. E.*]

8. The crank OP, Fig. 24, revolves about O, and by means of the slider at P causes the arm AB to rotate about A. The link BC drives

the slider C, which moves along an axis perpendicular to AO and distant 4 in. from O. OA is 10; OP, 3; AB, 16; and BC, 11 in. respectively.

If the crank OP makes 80 revs. per min., find the maximum velocity of the slider C and also its velocity when the angle AOP is 135°.

[*I. Mech. E.*]

9. The centre line SXS^1 for a fixed straight slotted bar is inclined to an axis OX at an angle of 30° and the distance OX is 18 in. A crank OC rotates about the fixed point O, and by means of a link CP drives the pin P which slides in the slotted bar SXS^1. The crank OC is 6 in. long, and when OC is at right angles to the axis OX the pin P is at the point X. Find the length of travel of the pin P and the maximum velocity of P, if the crank OC makes 120 revs. per min. [*I. Mech. E.*]

10. A quadrilateral $ABCD$ represents the centre line diagram of a four-bar kinematic chain in which AD is fixed and is 8 ft. long. The link AB rotates about A with uniform speed of 100 revs. per min., and by means of the link BC drives the link CD, which rotates about D. AB is 2 ft., BC 9 ft., and CD 4 ft. long respectively.

Determine the magnitude of the angle through which the link CD moves, and also find the angular speed of the links CD and BC in one of the positions in which the link BC is perpendicular to the link AB.

[*I. Mech. E.*]

11. In a four-bar link motion the links AB, BC, CD, and DA are 16, 12, 28, and 24 in. long respectively. The diameter of each pin is 2 in. Assuming that AB is fixed and that BC rotates at a speed of 80 revs. per min., determine the speed of rubbing at each pin when BC is perpendicular to BA. [*Inst. C. E.*]

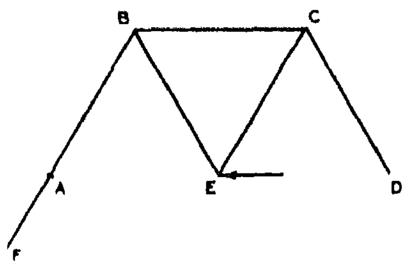


Fig. 25.

12. The link work $FBCD$, Fig. 25, has pin joints at A, B, C, and D, the points A and D being fixed in position. The connecting link BC carries a triangular frame BCE. The bars BC, CD, BE, and CE are each 8 in. in length; the length of FB is 11 in. and of AB 8 in.; the distance AD is 16 in. A horizontal force of 10 lb. acts at E as shown. Neglecting the effects of friction, determine the magnitude of the vertical force at F to maintain equilibrium. [*Inst. C. E.*]

13. The crank radius of an oscillating engine is 1 ft. long, and the shaft rotates at 90 revs. per min. The piston rod is 4 ft. long, and the distance between the centres of the crank-shaft and trunnions is 4 ft. Find the absolute velocity of the piston in magnitude and direction when the crank is at an angle of 60° from the outer dead centre. [*Lond. B.Sc.*]

14. A riveter, Fig. 26, is operated by a piston F acting through the links EB, AB, BC; D is the ram carrying the tool. The piston moves in a line perpendicular to the line of motion of D. $BC = 2AB$.

In the position shown, AB makes 12° with AC, and BE is at 90° to AC. Find the velocity-ratio of E to D. In the same position, the total load on the piston is 500 lb.; find the thrust exerted by D if the efficiency is 72 per cent. [*Lond. B.Sc.*]

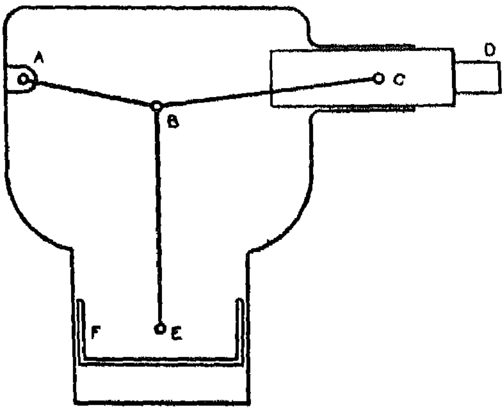


Fig. 26.

15. The crank of a reciprocating engine has a radius of 1 ft. and makes 100 revs. per min.; the connecting-rod is 4 ft. long; the crank pin is 3 in. diameter.

Determine, when the crank makes 45° from the inner dead centre: (*a*) the velocity of the piston; (*b*) the angular velocity of the connecting-rod; (*c*) the velocity of rubbing on the crank pin.

16. A rotary cylinder engine has a stroke of $4\frac{1}{2}$ in. and connecting-rods 9 in. long. Determine the instantaneous speeds of rubbing at the crank and gudgeon pins of the master connecting-rod when the latter is inclined at 45° to the crank and the cylinders are rotating at a constant speed of 1,200 revs. per min. The diameters of the crank pin and gudgeon are $1\frac{3}{4}$ in. and 1 in. respectively. [*Lond. B.Sc.*]

17. The outline of a portion of the mechanism of a Joy's valve gear is shown in Fig. 27.

The crank OR is 1 ft. and the connecting-rod CR 6 ft. 3 in. The length RD is 4 ft. AB is a swing link 3 ft. 8 in. long pivoting at the fixed point A. BD is 1 ft. $7\frac{1}{2}$ in. in length. The crank revolves at 210 revs. per min.

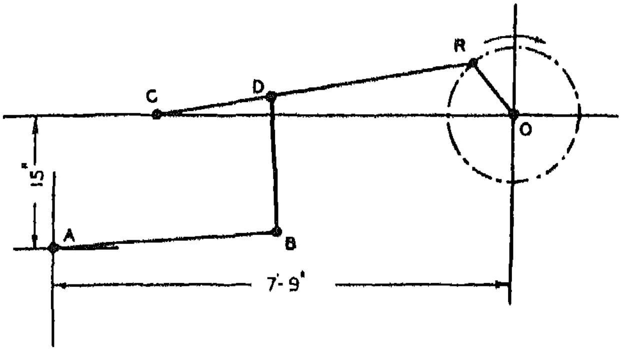


Fig. 27.

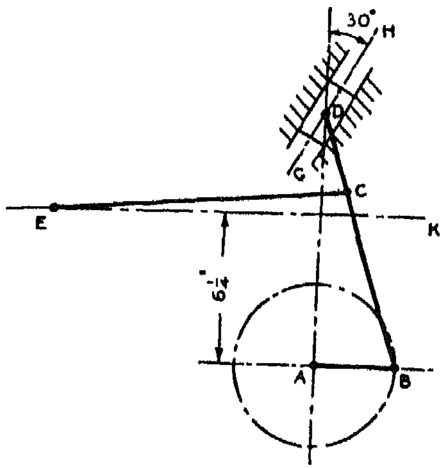


Fig. 28.

Find the angular velocity of the link BD at the instant when the crank angle ROC is 150°. [*Lond. B.Sc.*]

18. In the mechanism of the valve gear shown in Fig. 28, AB is 4 in.; BD, 16 in.; DC, 8 in.; and CE, 20 in. long. AB rotates at 150 revs. per min., and the point D is guided to move in the straight

line GH. E moves in the straight line EK. Find the velocity of E for the configuration shown.

19. The driving mechanism in a planing machine is as follows: A crank OP revolves in a vertical plane about the fixed point O, and the pin P slides in the slotted lever PQT, which turns about the fixed point Q, which is 8 in. vertically above the point O. The other end of the lever is slotted to take a pin T on the tool carrier, which moves in a horizontal plane. The horizontal plane of motion of the pin T is 9 in. above the point Q. The crank OP revolves 20 times per min. and can be adjusted to any length up to 5 in. Determine—

(*a*) The minimum length of slot in which the pin T slides.
(*b*) The radius of OP in order to give a tool travel of 6 in.
(*c*) The mean and maximum velocity of the tool during the 6-in. cutting-stroke.
(*d*) The actual velocity of the tool at one-quarter of the 6-in. cutting stroke. [*I. Mech. E.*]

20. A single cylinder petrol-engine has a stroke of 3 in. and length of connecting-rod $6\frac{1}{2}$ in.; the speed of the engine is 1,500 revs. per min. Find the angular velocity of the connecting-rod when the crank has turned through an angle of 45° from the inner dead centre.

What is the velocity of a point on the connecting-rod 2 in. from the crank pin when the crank is in the above position?

21. What do you understand by the term 'instantaneous centre'? Find the instantaneous centre for the motion of a connecting-rod which is four times as long as the crank when the crank is at an angle of 30° from the inner dead centre. If the crank is 4 ft. long, find the instantaneous velocity of the crosshead when the crank is turning at the uniform rate of 300 revs. per min. [*Lond. B.Sc.*]

22. In the mechanism shown in Fig. 29 the arms AB and CD are equal and when in its mid position (as shown) these arms are equally inclined to the horizontal. The coupler BC carries an arm EF at right angles BC and the length of EF is 3 inches.

When AB has moved 10 degrees to the left of its mid position its angular velocity is 10 radians per sec. Find the angular velocity of CD and the velocity of the point F. [*Lond. B.Sc.*]

23. Fig. 30 shows the arrangement of the crank and connecting-rods of each pair of cylinders in a multi-cylinder 60° Vee engine. Pistons are attached to the connecting-rods at C and E and the lines of stroke are along CA and EA. The crank is shown symmetrically between the lines of stroke. Find the velocities of the pistons at C and E for the given configuration for an engine speed of 2,000 revs. per min. [*Lond. B.Sc.*]

24. Fig. 31 shows a mechanism for obtaining a plunger travel of approximately four times the crank radius. The crank AB rotates

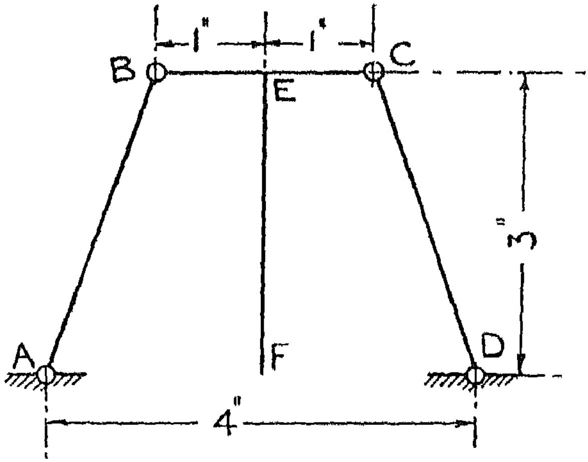


Fig. 29.

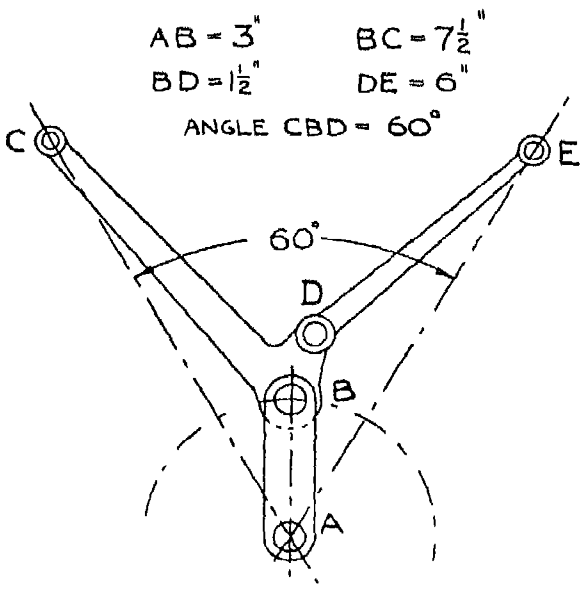


Fig. 30.

at a constant speed of 120 revs. per min. On the crank pin at B is a yoke or coupler CD. The end C is pin-jointed to a swinging link CF which can swing about the fixed centre F; the other end D of the coupler CD is pin-jointed to the connecting-rod DE. The end E drives the plunger whose motion is in a vertical path through E. For the given configuration find the velocity of E and the angular velocity of the coupler CD. [*Lond. B.Sc.*]

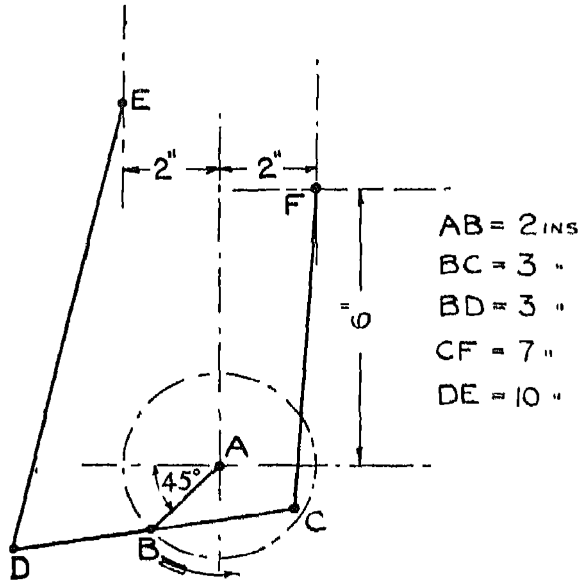
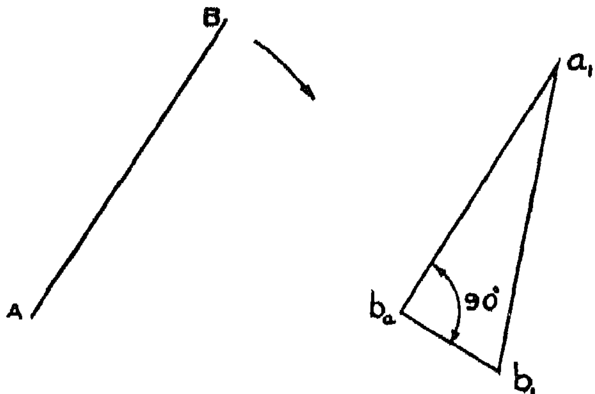


Fig. 31.

CHAPTER IV

ACCELERATION DIAGRAMS

§ **51. Total Acceleration.** A particle moving in a circular path of radius r ft., at an angular velocity of ω radians per sec., has, as already explained, a centripetal acceleration of $\omega^2 r$ ft. per sec. per sec. This acceleration may be expressed as $\frac{v^2}{r}$, since $v = r\omega$, where v is the tangential or peripheral



(a) (b)

Fig. 32.

speed. In Fig. 32 (a), let AB represent a link capable of rotating about A. Let v = velocity of B relative to A, then the centripetal acceleration of B relative to A is in a direction from B to A and is of magnitude $\frac{v^2}{AB}$. This acceleration, having direction and magnitude, may be represented by a vector $a_1 b_a$, Fig. 32 (b).

If the link AB does not rotate at uniform angular speed, it has an acceleration either positive or negative. If positive the speed is increasing, and if negative it is decreasing. Assume the direction of rotation clockwise as shown by the arrow, and the acceleration to be positive, i.e. the speed is increasing in the direction of the arrow.

Let α = angular acceleration in radians per sec. per sec.

Then tangential acceleration of $B = \alpha . AB$, since $a = r\alpha$. This acceleration may be represented by the vector $b_a b_1$. The point B has two accelerations, viz. centripetal and tangential, and the total or resultant acceleration of B relative to A is the vector sum of these accelerations and is represented by the vector $a_1 b_1$.

§ 52. **Notation for Acceleration Diagrams.** The notation used for acceleration diagrams is analogous to that used for velocity diagrams. Capital letters A, B, etc., will be used to denote points on the configuration diagram; small letters a, b, etc., will be used for the velocity diagram, as already explained, and small letters with a subscript for the acceleration diagram, such as a_1, b_1, a_b, b_a, etc. In contradistinction to the velocity diagram, in which a single vector completely represents the velocity of one point relative to another, the complete or total acceleration of one point relative to another requires two vectors, one to represent the centripetal acceleration and the other the tangential acceleration. The sum of these two vectors is the total acceleration and is represented by $a_1 b_1$, etc. The junction point of the two component vectors is represented by two letters—one a subscript to the other.

In Fig. 32 (b), $a_1 b_1$ is the total acceleration of B relative to A. The vector $a_1 b_a$ is the centripetal acceleration of B relative to A, and $b_a b_1$ is the tangential acceleration of B relative to A. Either b_a or a_b may be used to represent this junction point, the combination of the letters a and b merely indicating that relative acceleration between A and B is implied.

§ 53. **Method of Procedure.** The method to be adopted in the construction of acceleration diagrams is an extension of that already used for the construction of velocity diagrams. Assuming the configuration and velocity diagrams to be drawn, the first step is to calculate the centripetal accelerations; these are readily calculated by the aid of the velocity diagram. For a link rotating about one extremity the centripetal acceleration of the other extremity is the square of its relative velocity divided by the length of the link.

The construction of the acceleration diagram is greatly

facilitated by the use of the fundamental acceleration equation, similar in form to that used for velocity diagrams. Abbreviating centripetal to cent., tangential to tang., and acceleration to acc., we have—

acc. of C relative to A = acc. of C relative to B
+acc. of B relative to A,

or $$a_1 c_1 = b_1 c_1 + a_1 b_1 = a_1 b_1 + b_1 c_1.$$

The total acceleration, however, is composed of two component accelerations—centripetal and tangential. The above form of acceleration equation must therefore be modified to include these components. In its modified form the complete equation is

(cent.+tang.) acc. of C relative to A
= (cent.+tang.) acc. of C relative to B
+(cent.+tang.) acc. of B relative to A,

or $$a_1 c_a + c_a c_1 = (b_1 c_b + c_b c_1) + (a_1 b_a + b_a b_1)$$
$$= (a_1 b_a + b_a b_1) + (b_1 c_b + c_b c_1).$$

It is now possible to proceed with the construction of the acceleration diagram.

Before applying the foregoing method, there are two points which are of importance:

(1) A point moving in a straight path has no centripetal acceleration, since the point is virtually moving in the arc of a circle of infinite radius.

(2) A point at the end of a link which moves with constant angular velocity has no tangential acceleration.

§ 54. **Angular Acceleration of a Link.** When the tangential acceleration of one extremity of a link, relative to the other, is known the angular acceleration of the link is readily found by dividing the tangential acceleration by the length of the link, since $a = r.\alpha$.

§ 55. **Acceleration Diagram for Quadric Cycle Chain.** Let $A_1 BCA_2$, Fig. 15 (a), represent a quadric cycle chain and Fig. 15 (b) the velocity diagram for the given configuration at the instant B is rotating clockwise at a given speed.

Let α = angular acceleration of $A_1 B$, assumed positive, i.e. speed increasing.

Calculating the centripetal accelerations,

$$\text{cent. acc. of } C \text{ relative to } A = \frac{(ac)^2}{A_0 C},$$

$$C \quad ,, \quad ,, \quad B = \frac{(bc)^2}{BC},$$

$$B \quad ,, \quad ,, \quad A = \frac{(ab)^2}{A_1 B}$$

$$\text{tang.} \quad ,, \quad B \quad ,, \quad ,, \quad A = A_1 B.$$

The acceleration equation stated in § 53 is applicable to this problem and the acceleration diagram may now be drawn.

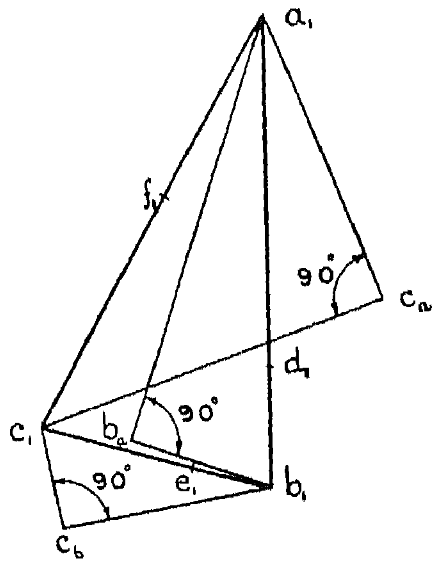


Fig. 33.

In Fig. 33, $a_1 b_a$ is drawn in a direction from B to A_1 and of magnitude proportional to $\frac{(ab)^2}{A_1 B}$. The vector $b_a b_1$ is drawn perpendicular to this and of magnitude proportional to $A_1 B . \alpha$. Joining a_1 and b_1, $a_1 b_1$ represents the total acceleration of B relative to A.

The vector $b_1 c_b$ is drawn to represent the centripetal acceleration of C relative to B, in a direction from C to B and

of magnitude proportional to $\frac{(bc)^2}{BC}$. The vector $c_b c_1$ is drawn perpendicular to this, but its magnitude is unknown and c_1 is not yet located.

The vector $a_1 c_a$ is drawn to represent the centripetal acceleration of C relative to A, in a direction from C to A_2 and of magnitude proportional to $\frac{(ac)^2}{A_2 C}$. The vector $c_a c_1$ is drawn perpendicular to this to represent the tangential acceleration of C relative to A. The intersection of $c_b c_1$ and $c_a c_1$ at c_1 locates c_1.

Joining $b_1 c_1$ and $a_1 c_1$, $a_1 c_1$ represents the total acceleration of C relative to A and $b_1 c_1$ the total acceleration of C relative to B. The angular acceleration of the link $A_2 C$ is $\frac{c_a c_1}{A_2 C}$ and of the link BC (about B), $\frac{c_b c_1}{BC}$.

In accordance with the scheme of notation, a_c could be used in place of c_a, and b_c for c_b.

The acceleration of points D, E, and F on the links $A_1 B$, BC, and $A_2 C$ respectively is obtained by dividing the vectors $a_1 b_1$, $b_1 c_1$, and $c_1 a_1$ in points d_1, e_1, and f_1 respectively such that

$$\frac{a_1 d_1}{a_1 b_1} = \frac{A_1 D}{A_1 B}; \qquad \frac{b_1 e_1}{b_1 c_1} = \frac{BE}{BC}; \qquad \frac{c_1 f_1}{c_1 a_1} = \frac{CF}{CA_2}.$$

§ 56. Acceleration Diagram for Slider Crank Chain. Let $A_1 BC$, Fig. 16 (a), represent a slider crank chain, and abc, Fig. 16 (b), the velocity diagram for the given configuration. Let the crank $A_1 B$ rotate at constant angular speed.

Calculating the centripetal accelerations,

$$\begin{array}{llll} \text{cent. acc. of } C & \text{relative to} & A = & 0, \\ \quad C & & B & \frac{(bc)^2}{BC}, \\ \quad B & & & \frac{(ab)^2}{A_1 B}, \\ \text{tang.} \quad B & & & 0. \end{array}$$

The acceleration equation stated in § 53 is applicable and the acceleration diagram may now be drawn.

In Fig. 34, $a_1 b_1$ is drawn in a direction from B to A_1 and of magnitude proportional to $\frac{(ab)^2}{A_1 B}$ to represent the centripetal acceleration of B relative to A. This vector represents also the total acceleration, since the tangential acceleration is zero.

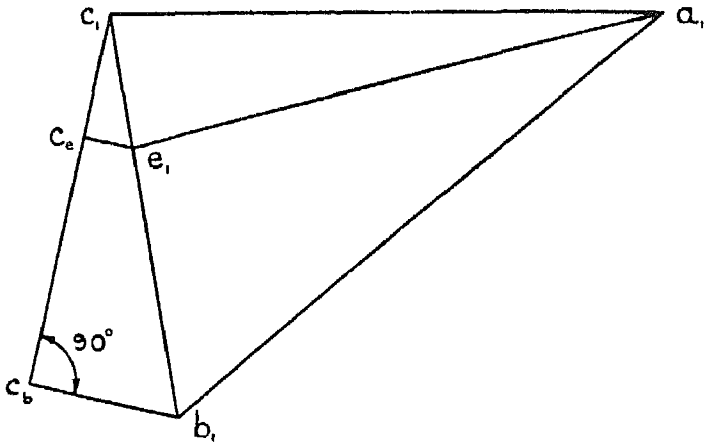


Fig. 34.

The vector $b_1 c_b$ is drawn to represent the centripetal acceleration of C relative to B, in a direction from C to B and of magnitude proportional to $\frac{(bc)^2}{BC}$. The vector $c_b c_1$ is drawn perpendicular to this, its magnitude is unknown and c_1 is not yet located.

The vector $a_1 c_1$ is drawn parallel to the line of stroke to represent the tangential acceleration of C relative to A and represents the total acceleration, since the centripetal acceleration of C relative to A is zero. The intersection of $c_b c_1$ and $a_1 c_1$ at c_1 locates c_1.

Joining $b_1 c_1$, the triangle $a_1 b_1 c_1$ is the total acceleration diagram and the tangential acceleration of C relative to A is represented by $a_1 c_1$. The angular acceleration of the connecting-rod BC is $\frac{c_b c_1}{BC}$.

The acceleration of E on BC is found by dividing $b_1 c_1$ at e_1 such that $\frac{b_1 e_1}{b_1 c_1} = \frac{BE}{BC}$.

Joining $a_1 e_1$, $a_1 e_1$ represents the total acceleration of E relative to A. The component accelerations of E are obtained by drawing $e_1 c_e$ parallel to $b_1 c_b$. The vector $e_1 c_e$ represents the centripetal acceleration of E and $c_e c_1$ the tangential acceleration.

§ 57. **Offset Point on a Link.** It frequently happens that a link has an offset point. In Fig. 35 (a), let AB represent a

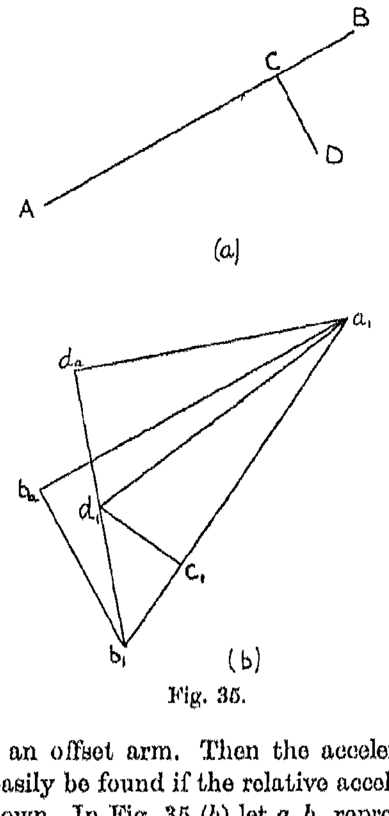


Fig. 35.

link and CD an offset arm. Then the acceleration of the point D can easily be found if the relative accelerations of A and B are known. In Fig. 35 (b) let $a_1 b_a$ represent the centripetal acceleration of B relative to A, $b_a b_1$ the tangential

acceleration of B relative to A, then $a_1 b_1$ is the complete acceleration. Since the tangential acceleration is $r\alpha$ and the centripetal acceleration is $r\omega^2$ or $\frac{v^2}{r}$, the acceleration of any point on a link depends upon the radius from the point of rotation. Hence the acceleration of the offset point D depends upon its distance from A. The centripetal acceleration of D is $\omega^2.AD$ and the tangential acceleration is $AD.\alpha$. These accelerations are represented by $a_1 d_a$ and $d_a d_1$ respectively drawn parallel to and perpendicular to AD. It is thus seen that the triangle $a_1 b_1 d_1$ is similar to the triangle ABD. Consequently if the relative accelerations of A and B are known, the acceleration of any offset point such as D is found by constructing the triangle $a_1 b_1 d_1$ similar to the triangle ABD.

EXAMPLE 1. In the mechanism shown in Fig. 36 (a) the connecting link CA_2 is perpendicular to A_1E when the point C falls on E. $A_1B = 9$ in.; $A_1E = 3$ ft.; $BC = 3$ ft.; $A_2C = 4$ ft.; and $A_2D = 1$ ft.

The crank A_1B has a speed of 80 revs. per min. Determine the velocity and acceleration of the point D when the crank angle BA_1E is 30°. [*Inst. C. E.*]

The velocity of D is known when the velocity of C is found.

Velocity of C relative to A = velocity of C relative to B + velocity of B relative to A;

or $$ac = bc + ab = ab + bc.$$

Velocity of B relative to $A = \frac{80 \times 2\pi}{60} \times \frac{9}{12} = 6{\cdot}28$ ft. per sec.

In Fig. 36 (b), ab is drawn perpendicular to A_1B and proportional to 6·28. The vector bc is drawn perpendicular to BC and ac perpendicular to A_2C. The vector ac represents the velocity of C and this scales 3·9 ft. per sec.

$\therefore$ velocity of $D = \frac{3{\cdot}9}{4} = 0{\cdot}975$ ft. per sec. The vector bc scales 6·1 ft. per sec.

Using the acceleration equation given in § 53 and calculating the centripetal accelerations:

cent. acc. of C relative to $A = \frac{(3{\cdot}9)^2}{4} = 3{\cdot}8$ ft. per sec. per sec.

" " C " " $B = \frac{(6{\cdot}1)^2}{3} = 12{\cdot}4$ " " "

cent. acc. of B relative to $A = \frac{(6 \cdot 28)^2}{0 \cdot 75} = 52 \cdot 6$ ft. per sec. per sec.

tang. ,, B ,, ,, $A = 0$.

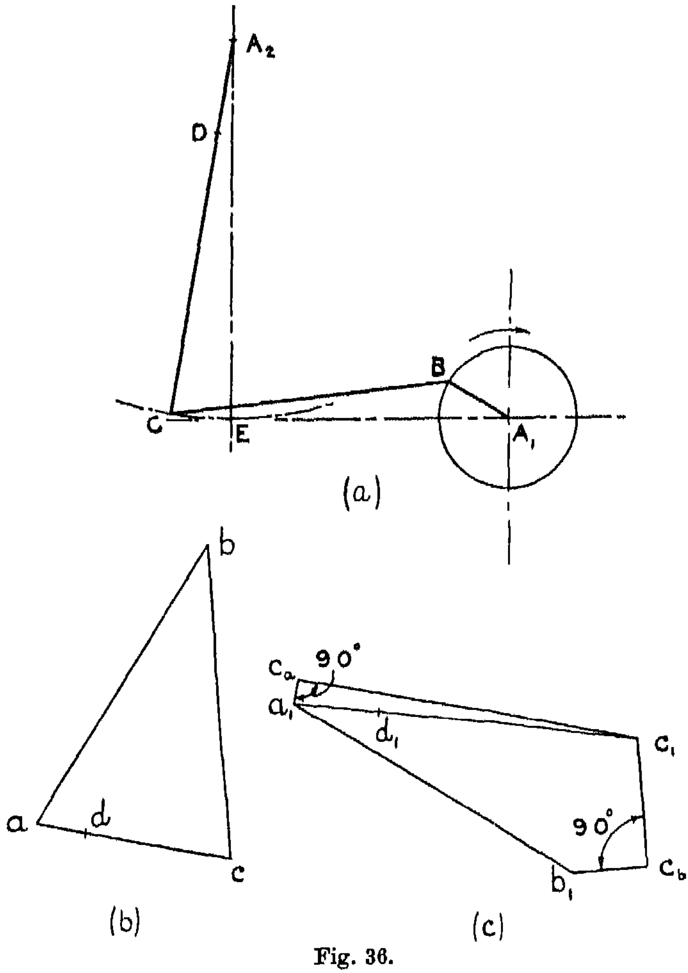


Fig. 36.

In Fig. 36 (c), $a_1 b_1$ is drawn in a direction from B to A_1, and of magnitude proportional to 52·6. This is the total acceleration of B relative to A, since $A_1 B$ rotates at constant speed.

The vector $b_1 c_b$ is drawn in a direction from C to B and of magnitude proportional to 12·4; this represents the centripetal

acceleration of C relative to B. Perpendicular to $b_1 c_b$, the vector $c_b c_1$ is drawn to represent the tangential acceleration of C relative to B, but this magnitude being unknown, the point c_1 is not yet located.

The vector $a_1 c_a$ is drawn proportional to 3·8 and in a direction from C to A_2; $c_a c_1$ is drawn perpendicular to this, to represent the tangential acceleration of C relative to A, intersecting $c_b c_1$ in c_1, thus locating the point c_1. The total acceleration of C relative to A is $a_1 c_1$ and this scales 56 ft. per sec. per sec. The acceleration of D is therefore $\frac{56}{4}$ or 14 ft. per sec. per sec. The point d_1 is shown on $a_1 c_1$ and is obtained by making $\dfrac{a_1 d_1}{a_1 c_1} = \dfrac{A_2 D}{A_2 C}$.

Example 2. The mechanism shown in Fig. 37 (a) is that for a valve gear. The crank $A_1 P$ rotates at constant speed of 12 radians per sec., and is pinned at P to the rod PR, the point Q in this rod being guided in a circular path by the oscillating link $A_2 Q$, the centre of oscillation being at A_2. The valve rod is connected at V to R by the coupler VR.

For the given configuration find the velocity and acceleration of the valve rod V.

$A_1 P = 5$; $PQ = 14$; $PR = 16{\cdot}5$; $RV = 12$; and $QA_2 = 17$ in.

Velocity of P relative to $A_1 = 12 \times \frac{5}{12} = 5$ ft. per sec.

Velocity of Q relative to A = velocity of Q relative to P + velocity of P relative to A;

or $$aq = pq + ap = ap + pq.$$

In Fig. 37 (b) ap is drawn perpendicular to $A_1 P$ and proportional to 5 ft. per sec. The vectors aq and pq are drawn perpendicular to $A_2 Q$ and PQ respectively. The point r is found by making
$$\frac{pr}{pq} = \frac{PR}{PQ}.$$

Velocity of V relative to A = velocity of V relative to R + velocity of R relative to A;

or $$av = rv + ar = ar + rv.$$

The vectors rv and av are drawn perpendicular to RV and parallel to the line of motion of V respectively. The vector av represents the velocity of V and this scales 0·75 ft. per sec.

Measuring the relative velocities and calculating the centripetal accelerations,

aq represents 3·7 ft. per sec. $\therefore$ cent. acc. of Q relative to A
$$= \frac{(3{\cdot}7)^2}{\frac{17}{12}} = 9{\cdot}67 \text{ ft. per sec. per sec.}$$

pq represents 2·05 ft. per sec. $\therefore$ cent. acc. of Q relative to P

$$= \frac{(2{\cdot}05)^2}{\frac{14}{12}} = 3{\cdot}6 \text{ ft. per sec. per sec.}$$

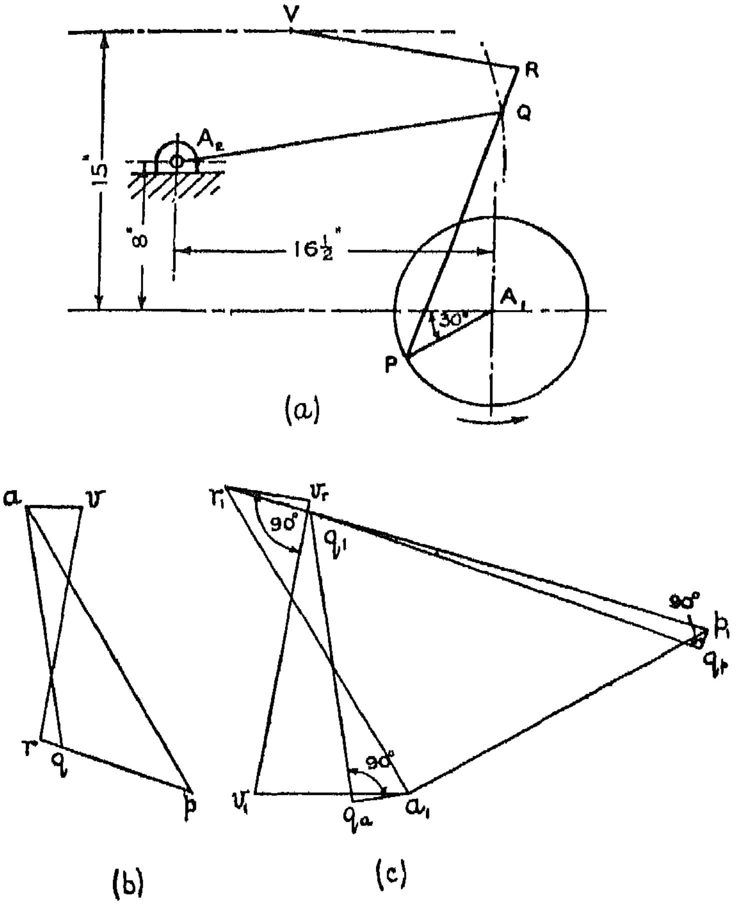


Fig. 37.

ap represents 5 ft. per sec. $\therefore$ cent. acc. of P relative to A

$$= \frac{(5)^2}{\frac{5}{12}} = 60 \text{ ft. per sec. per sec.}$$

Tang. acc. of P relative to $A = 0$.

(Cent. + tang.) acc. of Q relative to A = (cent. + tang.) acc. of Q relative to P + (cent. + tang.) acc. of P relative to A;

or
$$a_1 q_a + q_a q_1 = (p_1 q_p + q_p q_1) + (a_1 p_a + p_a p_1)$$
$$= (a_1 p_a + p_a p_1) + (p_1 q_p + q_p q_1).$$

In Fig. 37 (*c*), $a_1 p_1$ is drawn in a direction from P to A_1 and of magnitude proportional to 60; this vector is equal to $a_1 p_a + p_a p_1$ and since $p_a p_1 = 0$, p_1 coincides with p_a. The vector $p_1 q_p$ is drawn in a direction from Q to P and of magnitude proportional to 3·6; $q_p q_1$ is drawn perpendicular to this vector, but the magnitude is unknown and q_1 is not yet located.

The vector $a_1 q_a$ is drawn in a direction from Q to A_2 and of magnitude proportional to 9·67; $q_a q_1$ is drawn perpendicular to $A_2 Q$, meeting $q_p q_1$ in q_1.

Joining p_1 and q_1, $p_1 q_1$ represents the total acceleration of Q relative to P. The point r_1 is found by making $\frac{p_1 r_1}{p_1 q_1} = \frac{PR}{PQ}$.

The acceleration of V is obtained by forming a further acceleration equation between V, R, and A.

(Cent.+tang.) acc. of V relative to A = (cent.+tang.) acc. of V relative to R+(cent.+tang.) acc. of R relative to A.

The total acceleration of R relative to A is already found by joining $a_1 r_1$, hence it is only necessary to calculate the centripetal accelerations of V relative to A and of V relative to R.

Cent. acc. of V relative to $A = 0$.

rv represents 3·6 ft. per sec. $\therefore$ cent. acc. of V relative to R

$$= \frac{(3{\cdot}6)^2}{1} = 13 \text{ ft. per sec. per sec.}$$

The vector $r_1 v_r$ is drawn in a direction from V to R and of magnitude proportional to 13; $v_r v_1$ is drawn perpendicular to this vector. The vector $a_1 v_1$ is drawn parallel to the line of motion of V, meeting $v_r v_1$ in v_1. The vector $a_1 v_1$ represents the acceleration of V relative to A and scales 25·4 ft. per sec. per sec.

§ 58. **Accelerations in Quick-Return Motions.** The problem of drawing acceleration diagrams for mechanisms where sliding at the end of the crank is accompanied by rotation of the slotted lever becomes more difficult than for simple link mechanisms because of the complex nature of the acceleration at the junction of the crank and slotted lever. The velocity diagrams for such mechanisms can readily be drawn by considering a point on the slotted lever immediately underneath the crank pin for the given

configuration. In Fig. 18 (a), the point B can be interpreted as representing two points, a point B at the end of the crank $A_1 B$ and a point B' immediately underneath B but actually on $A_2 C$. Thus B will refer to the end of the crank $A_1 B$ and B' to a point on $A_2 C$ which for the given configuration is coincident with B. The vector ab' in Fig. 18 (b) represents the velocity of B' relative to A_2 and the angular velocity of $A_2 B'$ or $A_2 C$ is $\frac{ab'}{A_2 B'}$. The vector $b'b$ represents the velocity of sliding of B relative to B'.

When considering the acceleration of B' in Fig. 18 (a), the centripetal accelerations of B and B' relative to A_1 and A_2 can readily be calculated. The tangential acceleration of B relative to A_1 is zero if $A_1 B$ is rotating at constant speed and the centripetal acceleration of B' relative to B is zero. The tangential acceleration of B' relative to B is along $A_2 B$ but its magnitude is unknown. The tangential acceleration of B' relative to A_2 is due to two component variable velocities:

(1) tangential acceleration due to angular acceleration of $A_2 B$,

(2) tangential acceleration due to angular velocity of $A_2 B$ combined with velocity of B' along $A_2 B$ and of magnitude $2\dot{x}\dot{\phi}$.†

The acceleration due to (1) is known in direction but unknown in magnitude; the acceleration due to (2) is known in direction and magnitude.

The acceleration of the cutting tool E can be found if the angular velocity and angular acceleration of $A_2 C$ are known, since an acceleration diagram for the links EC and CA_2 can be drawn by the method already outlined.

To find the angular velocity and acceleration of the link $A_2 C$ the following method of mathematical analysis can be used. In Fig. 18 (a) let

$$\theta = \text{angle } A_2 A_1 B,$$
$$\phi = \text{angle } A_1 A_2 B,$$
$$r = A_1 B,$$
$$l = A_1 A_2.$$

† For explanation of this term and graphical solution see Appendix.

From the triangle $A_1 A_2 B$

$$\frac{A_1 B}{\sin A_1 A_2 B} = \frac{A_1 A_2}{\sin A_1 B A_2},$$

i.e. $$r \sin\{\pi-(\theta+\phi)\} = l \sin\phi,$$

or $$r \sin(\theta+\phi) = l \sin\phi.$$

Differentiating each side with respect to time and using the abbreviated notation of $\dot{\theta} = \frac{d\theta}{dt}$ and $\dot{\phi} = \frac{d\phi}{dt}$,

$$r \cos(\theta+\phi)(\dot{\theta}+\dot{\phi}) = l \cos\phi \,.\, \dot{\phi}. \tag{1}$$

Evaluating for $\dot{\phi}$,

$$\dot{\phi} = \frac{\dot{\theta} r \cos(\theta+\phi)}{l \cos\phi - r \cos(\theta+\phi)}.$$

$\dot{\theta} = \frac{d\theta}{dt}$ = rate of change of θ with respect to time and is thus equal to ω, the angular speed of $A_1 B$.

$\dot{\phi} = \frac{d\phi}{dt}$ is the angular velocity of $A_2 B'$ or $A_2 C$.

Hence the angular velocity of $A_2 C$ is

$$\dot{\phi} = \frac{\omega r \cos(\theta+\phi)}{l \cos\phi - r \cos(\theta+\phi)}.$$

Differentiating equation (1) with respect to time

$$-r(\dot{\theta}+\dot{\phi})^2 \sin(\theta+\phi) + r(\ddot{\theta}+\ddot{\phi})\cos(\theta+\phi) = -l\dot{\phi}^2 \sin\phi + l\ddot{\phi}\cos\phi. \tag{2}$$

$\ddot{\theta} = \frac{d\dot{\theta}}{dt}$ is the angular acceleration of the crank $A_1 B$ and if this is rotating at constant speed then $\ddot{\theta} = 0$.

$\ddot{\phi} = \frac{d\dot{\phi}}{dt}$ = angular acceleration of $A_2 B'$ or $A_2 C$.

Hence if $\ddot{\theta} = 0$ equation (2) becomes

$$-r(\dot{\theta}+\dot{\phi})^2 \sin(\theta+\phi) + r\ddot{\phi}\cos(\theta+\phi) = -l\dot{\phi}^2 \sin\phi + l\ddot{\phi}\cos\phi$$

and $$\ddot{\phi} = \frac{l\dot{\phi}^2 \sin\phi - r(\dot{\theta}+\dot{\phi})^2 \sin(\theta+\phi)}{l \cos\phi - r \cos(\theta+\phi)},$$

or writing ω for $\dot{\theta}$

$$\ddot{\phi} = \frac{l\dot{\phi}^2 \sin\phi - r(\omega+\dot{\phi})^2 \sin(\theta+\phi)}{l\cos\phi - r\cos(\theta+\phi)}.$$

Values of $\dot{\phi}$ and $\ddot{\phi}$ can thus be calculated for any given configuration and from these values the acceleration diagram for $A_2 C$ and CE can be drawn.

Care must be exercised in the interpretation of signs when using $\dot{\phi}$ and $\ddot{\phi}$. The angle θ is measured from $A_1 A_2$ in a counter-clockwise direction, and ϕ is measured from $A_1 A_2$ in a clockwise direction. Hence a positive value for $\dot{\phi}$ indicates that $A_2 C$ is rotating in a clockwise direction and a negative value for $\dot{\phi}$ that $A_2 C$ is rotating counter-clockwise. Similarly, a positive value for $\ddot{\phi}$ means an increasing angular velocity in a clockwise direction.

The acceleration of sliding of the point B relative to B' can be found by obtaining an expression for the displacement of B from its top position F, and differentiating twice with respect to time. By drawing a perpendicular from A_1 on to $A_2 B$,

$$\text{the length } A_2 B = l\cos\phi + r\cos\{\pi - (\theta+\phi)\}$$
$$= l\cos\phi - r\cos(\theta+\phi).$$

Also $$A_2 F = l + r,$$

and the displacement of B along the lever is $A_2 F - A_2 B$; call this displacement x. Then

$$x = A_2 F - A_2 B$$
$$= l + r - \{l\cos\phi - r\cos(\theta+\phi)\}.$$

Differentiating with respect to time

$$v = \frac{dx}{dt} = l\dot{\phi}\sin\phi - r(\dot{\theta}+\dot{\phi})\sin(\theta+\phi).$$

Differentiating again with respect to time

$$a = \frac{dv}{dt} = l\dot{\phi}^2\cos\phi + l\ddot{\phi}\sin\phi - r(\dot{\theta}+\dot{\phi})^2\cos(\theta+\phi) - r(\ddot{\theta}+\ddot{\phi})\sin(\theta+\phi).$$

Substituting values of ω for $\dot{\theta}$, $\ddot{\theta} = 0$ for constant angular velocity of $A_1 B$, and for values of $\dot{\phi}$ and $\ddot{\phi}$, the acceleration of sliding can be found.

EXERCISES. IV

1. A locomotive having driving-wheels 6 ft. 6 in. in diameter has a linear acceleration of 0·8 ft. per sec. per sec. and an instantaneous velocity of 15 miles per hour. Assuming no slip of the driving-wheels, find the instantaneous velocity and acceleration of a point on the rim of the driving-wheel level with and in front of the wheel axis. [*I. Mech. E.*]

2. The stroke of a steam engine is 27 in., and the connecting-rod is 5 ft. long. Determine the velocity and acceleration of a point on the connecting-rod 2 ft. from the crank pin centre when the speed of the engine is 180 revs. per min. and the piston is moving towards the crank and is at quarter stroke. [*I. Mech. E.*]

3. In a quadric cycle chain $ABCD$, AD is the fixed link and AB rotates uniformly at 10 revs. per min. For the configuration in which AB is inclined at 60° to AD, determine: (*a*) the angular velocity of CD; and (*b*) the angular acceleration of CD.
$AD = 5$; $AB = 2\frac{1}{2}$; $BC = 3$; and $CD = 3$ in.

4. For a particular configuration, a link AB, 3 ft. long, is in a horizontal position. The point A is moving in a direction AD, 60° to AB with a velocity of 12 ft. per sec. and an acceleration of 50 ft. per sec. per sec. in the same direction. The end B of the link is constrained to move in a line BD, 45° to AB. The points ABD form a triangle, the base angles at A and B being 60° and 45° respectively.
Find the acceleration of B and of a point C in AB 1 ft. from B.

5. Show how to find the angular acceleration of the connecting-rod of a direct-acting engine for any particular configuration of the mechanism.

6. A trunnion engine has a crank radius of 18 in. The distance between the centres of the trunnion axis and the crank-shaft is 4 ft. If the crank rotates at 90 revs. per min., find the acceleration of the piston relative to the cylinder when the crank has turned through an angle of 135° from the inner dead centre. [*Lond. B.Sc.*]

7. A crank OC of 9 in. radius rotates about the fixed centre O with a uniform speed of 150 revs. per min. A connecting-rod CE, 35 in. long, drives a crosshead E, and the latter moves in a horizontal path which is $4\frac{1}{2}$ in. vertically above O. When the crank has turned through an angle of 60° from its inner horizontal position and taking the case when E is on the left side of O, find: (*a*) the length of the path traversed by E; (*b*) the velocity of E; (*c*) the acceleration of E.

8. The link AB of a mechanism shown in Fig. 38 rotates uniformly at 150 revs. per min. Find the velocity and acceleration of the slider F when AB is in the position shown. The rod CE is pivoted at D.
$EF = 5$; $AB = 2$; $BC = 4$; $CD = 2\frac{1}{2}$; $DE = 1$; and $AD = 4$ ft.
[*Lond. B.Sc.*]

9. Draw the velocity and acceleration diagrams for a steam-engine mechanism when the crank has turned 30° from the inner dead centre. Length of crank, 1 ft.; connecting-rod, 3 ft.; speed of crank pin, 10 ft. per sec. State the velocity and acceleration of the piston. [*Lond. B.Sc.*]

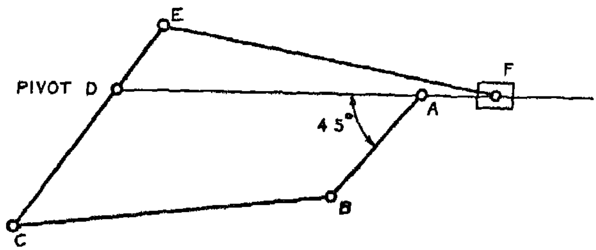


Fig. 38.

10. In the mechanism shown in outline in Fig. 39 the crank pin *A* moves uniformly in a circle with a velocity of 15 ft. per sec. Determine the acceleration of the slide block *B* for the position of the gear shown. [*Lond. B.Sc.*]

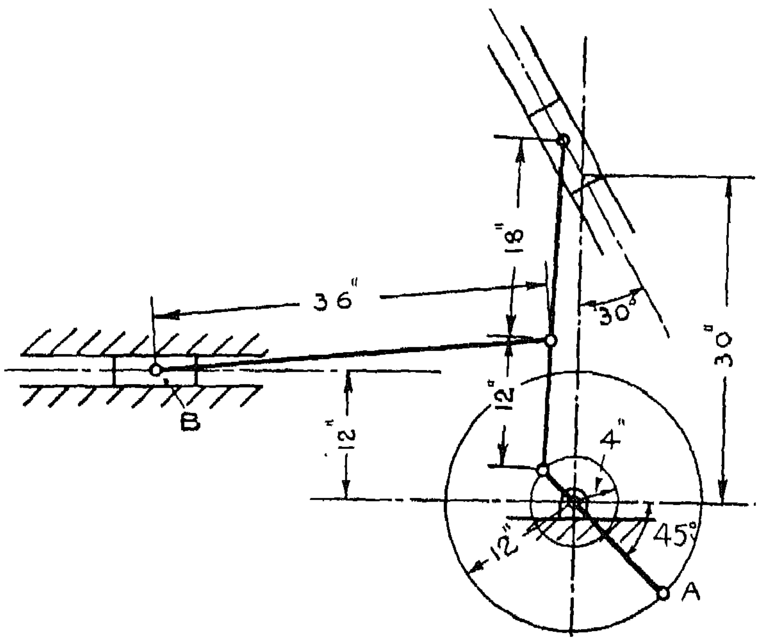


Fig. 39.

11. In a four-bar chain motion, Fig. 40, the lengths of the links a, b, c, and d are 4, 3, 7, and 6 in. respectively. Assuming that a is fixed and b is rotating at a speed of 2 revs. per sec., determine the instantaneous angular accelerations of c and of d for the given position of the chain. [*Lond. B.Sc.*]

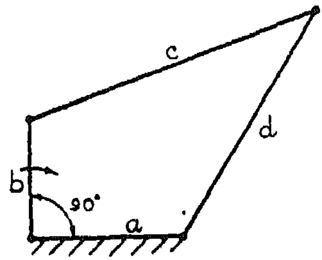


Fig. 40.

12. In the mechanism shown in Fig. 41, A and C are fixed centres, and AB moves with an angular speed of 12 radians per sec. For the given configuration, find: (a) the velocity of E; (b) the velocity of sliding in the block D; (c) the acceleration of sliding in the block D.
$AB = 2$; $AC = 3$; and $BE = 7$ in.

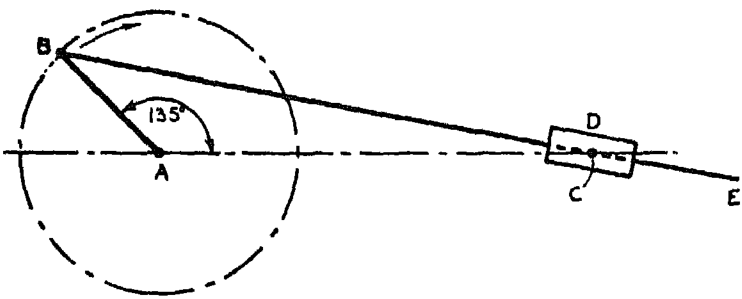


Fig. 41.

13. In the Whitworth quick-return motion shown in Fig. 42 the link AC rotates at a constant speed of 50 revs. per min. For the given configuration, find: (a) the ratio of the time of cutting to the time of return; (b) velocity of E; (c) acceleration of E.
$AC = 3$; $AB = 1\frac{1}{2}$; $BD = 3$; and $DE = 18$ in.

14. The diagram of a quick-return motion is shown in Fig. 43. The line of stroke of the pin H is along BH, which is perpendicular to AB. Find the length of the stroke and the time occupied by the cutting and return strokes respectively. The crank BC rotates uniformly at 60 revs. per min.

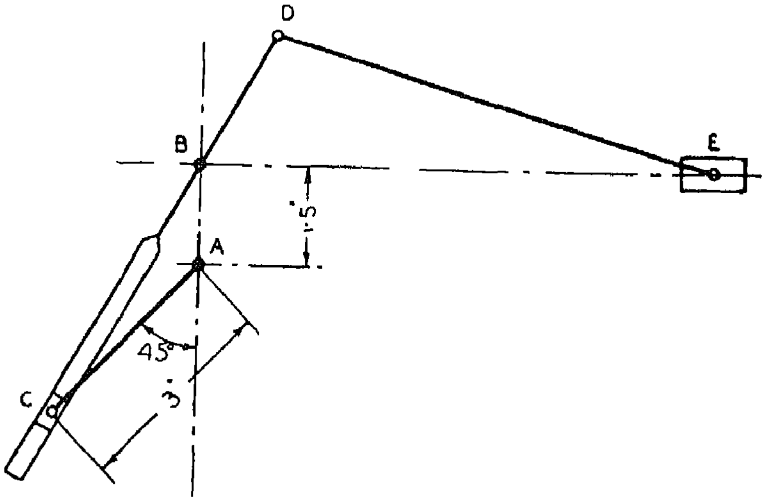


Fig. 42.

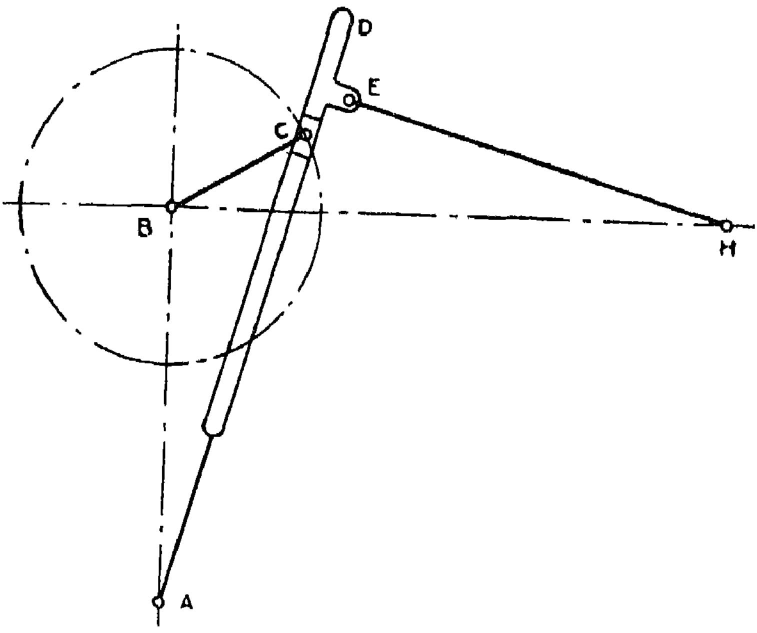


Fig. 43.

Find the velocity and acceleration of H when the angle HBC is 30°. $AB = 12$; $BC = 6$; $AD = 19\frac{1}{2}$; $AE = 15\frac{3}{4}$; $DE = 4\frac{1}{2}$; and $EH = 18$ in.

15. The crank radius of an oscillating engine is 12 in. The trunnion axis of the cylinder is 36 in. from the centre of the crank. The piston is 54 in. from the crank pin. When the crank makes an angle of 45° with the line joining the centre of the crank with the centre of the trunnion axis, and the crank pin is moving with a uniform velocity of 4 ft. per sec., find the velocity of the piston and angular acceleration of the cylinder. [*Lond. B.Sc.*]

CHAPTER V

SLIDER CRANK CHAIN

Diagrams of Displacement, Velocity, and Acceleration: Geometrical and Analytical Methods.

§ 59. **Piston Displacement.** The slider crank chain is of basic importance and as such requires rather more detailed attention than has been given in previous chapters, in which the general problems of motion have been dealt with. The necessity for finding the acceleration of moving parts arises because of the dynamical forces involved causing these accelerations. At high speeds these dynamical forces are of great importance.

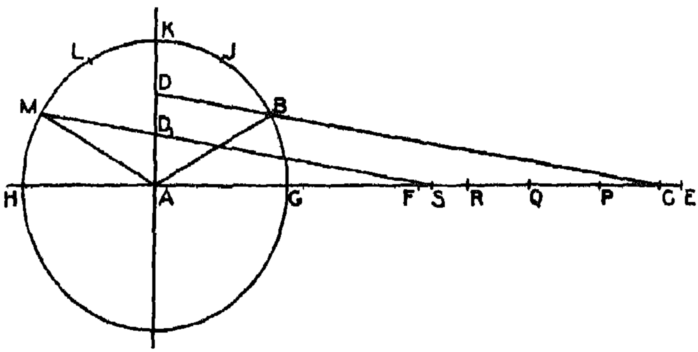


Fig. 44.

Referring to Fig. 44, let ABC represent a slider crank chain, in which AB represents the crank, BC the connecting-rod. The motion of C is that of the crosshead, to which is rigidly attached the piston. Let G and H represent the inner and outer dead centres respectively. The extreme positions of the crosshead are readily obtained by striking radii GE and HF, equal to the length of the connecting-rod, on the line of stroke. The points E and F represent the ends of the stroke. The displacement of the piston or crosshead for any crank position AB is obviously EC. The displacement of the piston for any other crank position, such as AJ, is obtained by striking off a circular arc from J, cutting EF in P.

§ 60. **Rectangular Diagram of Piston Displacement.** The relation which exists between the crank angle turned through from the inner dead centre and the piston displacement may be shown graphically by plotting values of the piston displacement against corresponding crank angles. In Fig. 44, let B, J, K, L, M, etc., be crank angles at 30° intervals: then the corresponding positions of the crosshead are C, P, Q, R, S, etc., obtained in the manner just described. The piston displacements corresponding to 30° crank intervals are EC, EP, EQ, ER, ES, etc.

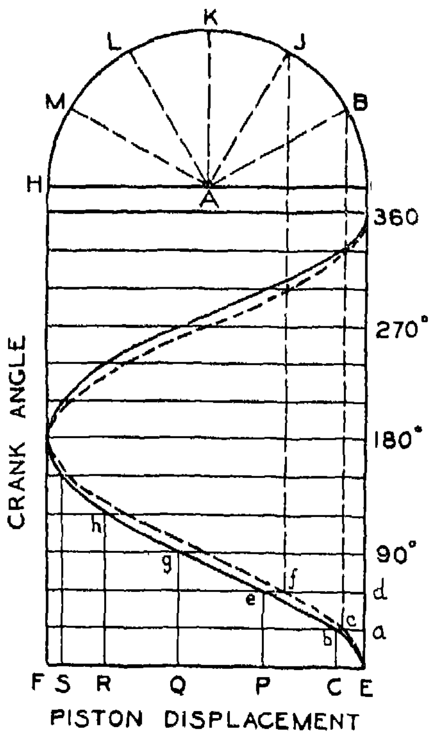


Fig. 45.

In Fig. 45, let EF represent the stroke, EC, EP, EQ, etc., piston displacements for 30° crank angle intervals. Taking vertical ordinates to represent crank angles, equal intervals (to any convenient scale) Ea, ad, etc., can each represent 30°.

Projecting up from C to meet the horizontal through a, the point b is obtained; in a similar manner, projecting up from P to meet the horizontal through d, the point e is obtained. Proceeding in this manner points g, h, etc., are obtained, and a smooth curve drawn through b, e, g, h, etc., represents graphically the piston displacement for any given crank angle.

The effect of the connecting-rod can be seen by comparing the diagram thus obtained with that for simple harmonic motion. The diagram of piston displacement for simple harmonic motion can be drawn by constructing a semicircle on the piston displacement. For convenience this is shown at the top of the diagram Fig. 45. Let B, J, K, etc., represent 30° crank intervals. Projecting down from B, to meet the horizontal through a, the point c is obtained. The point f is similarly found by projecting down from J to meet the horizontal through d. Proceeding in this manner and drawing a smooth curve through the points thus obtained, the curve for simple harmonic motion is obtained. This is shown dotted in Fig. 45. Diagrams plotted in the manner just described are called rectangular diagrams. Polar diagrams may be plotted by plotting the piston displacement along the corresponding crank position, but as these have no particular advantage over rectangular diagrams they need not be discussed further.

§ **61. Diagrams of Velocity of Piston.** The velocity of the piston or crosshead may be readily obtained by the method already described in § 43. In Fig. 44, for any crank position AB produce the connecting-rod to meet the vertical through A in D. Then, providing the angular velocity of the crank is uniform, the velocity of the piston is $\omega . AD$, where ω is the constant angular velocity of the crank. In the case where the connecting-rod cuts the vertical through A, as for the crank position AM, the intercept AD_1 is a measure of the piston velocity. The actual velocity is $\omega . AD_1$. The lengths AD and AD_1 are measured to the same scale that AB represents the length of the crank.

The values of the velocity of the piston plotted against

corresponding piston displacements give a continuous curve showing how the velocity of the piston varies throughout the stroke. In Fig. 46, let EF represent the stroke and EC, EP,

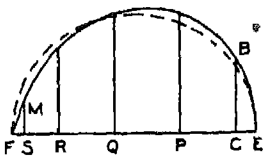


Fig. 46.

EQ, etc., piston displacements for 30° intervals of the crank. At C, the ordinate CB is made proportional to $\omega . AD$ (Fig. 44) or more simply, equal to AD, since ω is constant. Similarly SM is made equal to AD_1. Proceeding in this manner, points are obtained through which a smooth curve can be drawn. In Fig. 46 the variation of velocity is shown for one stroke only, as the crosshead moves from E to F; for the other stroke a diagram similar to that shown, but on the other side of EF, would be obtained.

The curve of velocity for simple harmonic motion is a semi-circle (for one stroke), shown dotted. The difference between the two curves shows clearly the effect of the connecting-rod. The greater the length of the connecting-rod in comparison with the crank, the more nearly will the actual curve approximate to that of simple harmonic motion. The difference between the two curves is due to the obliquity of the connecting-rod, and this difference becomes more marked for short connecting-rods than for long ones for a given length of crank.

A polar diagram may be drawn by setting off radially along the crank the corresponding velocity of the piston. The velocities may be either set off from the crank-shaft or the crank pin.

§ 62. **Acceleration of Piston.** The problem of finding the acceleration of the piston of the slider crank chain is of great importance on account of the dynamical forces involved, particularly in high-speed engines. The method already described, of finding the acceleration of the piston by drawing

the velocity and acceleration diagrams, becomes very laborious when the acceleration is required for several crank positions. In consequence, several ingenious geometrical constructions have been devised for rapidly determining the acceleration of the piston for any required crank angle. In each case of a geometrical construction, an acceleration diagram, similar to $a_1 b_1 c_b c_1$, Fig. 34, is obtained, without the troublesome necessity of calculating centripetal accelerations and of forming acceleration equations.

The better known constructions are those of Klein, Ritterhaus, and Bennett, and these will be described in the above order. These constructions only apply when the angular velocity of the crank is uniform.

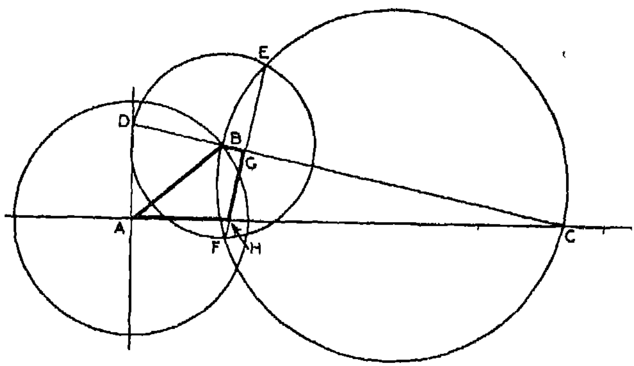


Fig. 47.

§ 63. **Klein's Construction.** In Fig. 47, let ABC represent a slider crank chain for a given crank position AB. The connecting-rod CB is produced to meet the vertical through A at D. With centre B and radius BD a circle DEF is described. A larger circle EBF is drawn on BC as diameter cutting the smaller circle in E and F. The points E and F are joined and produced if necessary to cut AC in H. Then the acceleration of the piston for the given configuration is $\omega^2 . AH$, where ω is the constant angular velocity of the crank and AH is measured to the same scale that AB represents the crank radius.

The figure $AHGB$ should be carefully compared with the

acceleration diagram of Fig. 34, in which $a_1 c_1$ is parallel to AH, $a_1 b_1$ is parallel to AB, $b_1 c_b$ is parallel to BG, and $c_b c_1$ parallel to GH.

The truth of Klein's construction is proved as follows: comparing the triangles DBA and abc of Fig. 16 (b), for equal crank angles, these triangles are similar; and since $\omega . AD = ac$,

$\therefore \ \omega . AB = ab$ and $\omega . BD = bc$.

In Fig. 47, if CE and EB be drawn, the triangles CEB and EGB are similar,

$$\therefore \frac{BG}{BE} = \frac{BE}{BC}. \quad \text{Hence } BG = \frac{BE^2}{BC} = \frac{BD^2}{BC} = \frac{(bc)^2}{\omega^2 . BC},$$

$\therefore \ BG . \omega^2 = \dfrac{(bc)^2}{BC}$, which is the centripetal acceleration of C relative to B.

Thus BG represents to scale the centripetal acceleration of C relative to B.

Again $\quad AB . \omega = ab \quad$ or $\quad AB^2 . \omega^2 = (ab)^2$,

$\therefore \ AB . \omega^2 = \dfrac{(ab)^2}{AB}$, which is the centripetal acceleration of B relative to A. The figure $ABGH$ is thus a reproduction of the acceleration diagram of Fig. 34.

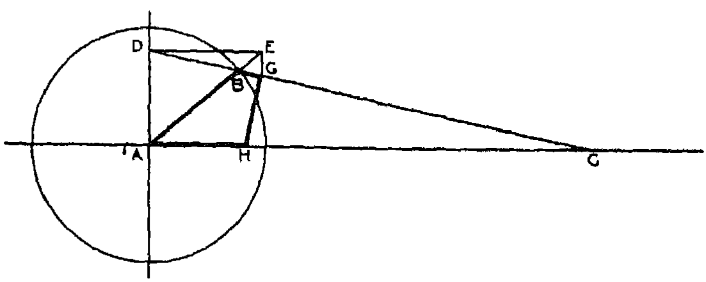


Fig. 48.

§ 64. **Ritterhaus's Construction.** In Fig. 48, CB is produced to meet the vertical through A in D. Through D, DE is drawn parallel to AC to meet AB produced in E. From E, EG is drawn parallel to AD to meet BC in G, and from G, GH is drawn perpendicular to BC, cutting AC in H. The acceleration of the piston is again $\omega^2 . AH$.

Ritterhaus's construction may be proved in the following manner. The triangles ABD, Fig. 48, and abc, Fig. 16 (*b*), are similar for equal crank angles; hence

$$\omega . AD = ac;\ \omega . AB = ab;\ \text{and}\ \omega . BD = bc.$$

In Fig. 48, the triangles BEG and ADB are similar,

and $$\frac{BD}{AD} = \frac{BG}{EG}. \qquad BG = BD . \frac{EG}{AD}.$$

Now $\frac{EG}{AD} = \frac{DE}{AC}$, since the triangles DEG and ADC are similar, and $\frac{DE}{AC} = \frac{BD}{BC}$, since the triangles DEB and ABC are similar.

Hence $$\frac{EG}{AD} = \frac{DE}{AC} = \frac{BD}{BC},$$

$$\therefore\ BG = BD . \frac{BD}{BC} = \frac{BD^2}{BC} = \frac{(bc)^2}{\omega^2 . BC},$$

whence $\omega^2 . BG = \frac{(bc)^2}{BC}$ = centripetal acceleration of C relative to B.

In a similar manner $AB . \omega^2$ may be shown equal to $\frac{(ab)^2}{AB}$, which is the centripetal acceleration of B relative to A.

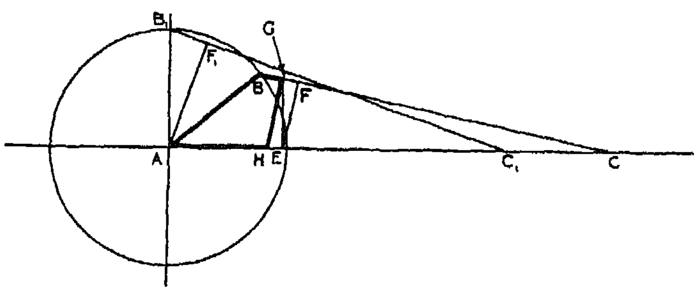


Fig. 49.

§ **65. Bennett's Construction.** A preliminary construction is required to find a point F on the connecting-rod. In Fig. 49, the crank AB_1 is drawn in a position perpendicular to the line of stroke: the corresponding crosshead position

is C_1. A perpendicular drawn from A on to $B_1 C_1$ locates the point F_1.

For any crank position AB, let F be the point corresponding to F_1 on $B_1 C_1$. From F, FE is drawn perpendicular to the connecting-rod BC to meet the line of stroke at E. From E, EG is drawn perpendicular to AC to meet BC in G, and from G, GH is drawn parallel to FE. The acceleration of the piston is $\omega^2 . AH$.

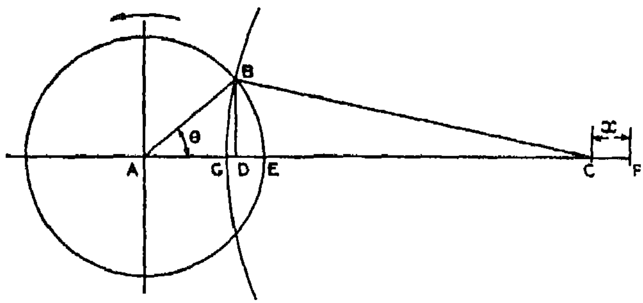


Fig. 50.

§ 66. **Approximate Analytical Method.** In Fig. 50, let ABC represent a slider crank chain, in which AB is the crank. Assuming the direction of rotation as shown by the arrow, let $\theta =$ angle turned through by the crank from the inner dead centre position AE. The corresponding cross-head positions are C and F, hence the displacement of the piston from the end position is FC; call this distance x. From B a circular arc BG is drawn, whose radius is equal to the length of the connecting-rod (i.e. with the centre C), to cut the line of stroke in G. From B a line BD, perpendicular to the line of stroke, is drawn, cutting AC in D.

Let $r =$ radius of crank AB,

$l =$ length of connecting-rod BC.

$$CB = FE = CG.$$

Then
$$\begin{aligned} x &= FC = FG - GC = FE + EG - GC = EG, \\ &= ED + DG, \\ &= AE - AD + DG, \\ &= r - r\cos\theta + DG. \end{aligned}$$

Now DG is part of the diameter of the circle of which BG is part, and BD is a chord perpendicular to this diameter. The diameter of the circle is $2l$ and the two parts of the diameter are GD and $(2l-GD)$. It is a well-known property of the circle that the square on half the chord, bisected by a diameter, is equal to the product of the two parts of the diameter.

In this case, $$BD^2 = GD(2l-GD),$$

or $$BD^2 = GD.2l-GD^2.$$

The square of GD is small in comparison with $GD.2l$, and if GD^2 is neglected a compact expression for velocity and acceleration of the piston is obtained.

Neglecting the square of GD,

$$BD^2 = GD\times 2l,$$

or $$DG = \frac{BD^2}{2l} = \frac{(r\sin\theta)^2}{2l},$$

$$\therefore\ x = r-r\cos\theta+\frac{r^2\sin^2\theta}{2l}. \tag{1}$$

Since rate of change of displacement is velocity, this expression may be differentiated with respect to t, to give the velocity of the piston:

$$\frac{dx}{dt} = 0-r(-\sin\theta)\frac{d\theta}{dt}+\frac{r^2}{2l}2\sin\theta\cos\theta\frac{d\theta}{dt}$$

$$= r\sin\theta\frac{d\theta}{dt}+\frac{r^2}{2l}\sin 2\theta\frac{d\theta}{dt}.$$

$\frac{d\theta}{dt}$ is rate of change of angular position, which is angular velocity ω. Writing ω for $\frac{d\theta}{dt}$ and v for $\frac{dx}{dt}$,

$$v = \omega r\left(\sin\theta+\frac{r}{2l}\sin 2\theta\right). \tag{2}$$

The rate of change of velocity is acceleration, and hence the differentiation of this expression with respect to t gives the acceleration of the piston:

$$\frac{dv}{dt} = \omega r\left(\cos\theta\frac{d\theta}{dt}+\frac{r}{2l}2\cos 2\theta\frac{d\theta}{dt}\right).$$

Writing a for $\frac{dv}{dt}$ and ω for $\frac{d\theta}{dt}$,

$$a = \omega^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right). \qquad (3)$$

The above equations (1), (2), and (3) for the displacement, velocity, and acceleration of the piston are only approximate, but sufficiently near the true values as to be very widely used, and, indeed, preferably used to the more cumbersome exact expressions. The exact expressions for velocity and acceleration are rarely used and differ so little from the approximate expressions that no serious error is introduced by the use of these approximate expressions. It should be remembered that although the acceleration diagrams and geometrical constructions already explained are correct in principle, they are only accurate as far as the accuracy of drawing will permit.

§ 67. **Exact Expression for Piston Acceleration.** The expression for piston acceleration given by equation (3), Art. 66 is only approximate. An exact expression can be derived as follows. Referring to Fig. 50 and using the same notation as in Art. 66,

$$\text{displacement of piston} = x = EG$$
$$= r - r\cos\theta + DG.$$

Now
$$DG = CG - CD$$
$$= l - \sqrt{(l^2 - r^2\sin^2\theta)},$$

hence
$$x = r - r\cos\theta + l - \sqrt{(l^2 - r^2\sin^2\theta)}.$$

Differentiating this expression with respect to t,

$$v = \frac{dx}{dt} = 0 + r\sin\theta\frac{d\theta}{dt} + \frac{1}{2}\frac{r^2 2\sin\theta\cos\theta}{(l^2 - r^2\sin^2\theta)^{\frac{1}{2}}}\frac{d\theta}{dt}.$$

Writing ω for $\frac{d\theta}{dt}$,

$$v = \omega r\left(\sin\theta + \frac{r\sin 2\theta}{2(l^2 - r^2\sin^2\theta)^{\frac{1}{2}}}\right).$$

Differentiating this expression with respect to t,

$$a=\frac{dv}{dt}=\omega r\left(\cos\theta\frac{d\theta}{dt}\right.$$

$$\left.+\frac{r}{2}\frac{\left[2\cos 2\theta(l^2-r^2\sin^2\theta)^{\frac{1}{2}}-\sin 2\theta\,.\,\frac{1}{2}\,\frac{-r^2 2\sin\theta\cos\theta}{(l^2-r^2\sin^2\theta)^{\frac{1}{2}}}\right]\frac{d\theta}{dt}}{(l^2-r^2\sin^2\theta)}\right)$$

$$=\omega^2 r\left(\cos\theta+\frac{rl^2\cos 2\theta+r^3\sin^4\theta}{(l^2-r^2\sin^2\theta)^{\frac{3}{2}}}\right).$$

This correct expression for piston acceleration differs only slightly from the approximate expression given in Art. 66.

If $\frac{l}{r}=n$, the exact expression becomes

$$a=\omega^2 r\left(\cos\theta+\frac{\frac{\cos 2\theta}{n}+\frac{\sin^4\theta}{n^3}}{\left(1-\frac{\sin^2\theta}{n^2}\right)^{\frac{3}{2}}}\right)$$

Taking a value of $n=4$, values of the acceleration of the piston are shown in the following table both for the approximate and correct values.

θ *deg.*	*Approx. acc.* $\times\,\omega^2 r$	*Correct acc.* $\times\,\omega^2 r$
0	1·250	1·250
20	1·131	1·134
40	0·809	0·814
60	0·375	0·375
80	−0·061	−0·068
100	−0·408	−0·415
120	−0·625	−0·625
140	−0·723	−0·718
160	−0·748	−0·746
180	−0·75	−0·75

Comparing the second and third columns in the above table it will be seen that they agree very closely and that the greatest difference occurs at comparatively small values of the acceleration, i.e. when θ is between 80° and 100°. The greatest values, when $\theta=0°$ and 180°, agree exactly.

§ **68. Diagram of Piston Acceleration.** The approximate expression given in § 66 for the acceleration of the piston may be conveniently used for finding the values of the acceleration for different values of the crank angle.

When

$\theta = 0°$, $\cos\theta = 1$, $\cos 2\theta = 1$, and $a = \omega^2 r\left(1 + \frac{r}{l}\right)$,

$\theta = 90°$, $\cos\theta = 0$, $\cos 2\theta = -1$, and $a = \omega^2 r\left(-\frac{r}{l}\right)$,

$\theta = 180°$, $\cos\theta = -1$, $\cos 2\theta = +1$, and $a = \omega^2 r\left(-1 + \frac{r}{l}\right)$.

Since values of $\frac{r}{l}$ are less than unity, the acceleration changes from positive to negative as θ varies from 0° to 180°. Plotting values of the acceleration obtained in this manner against corresponding piston displacements, a curve of piston acceleration is obtained as shown in Fig. 51.

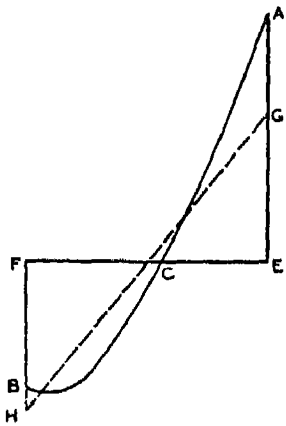


Fig. 51.

In Fig. 51, let EF represent the stroke or total piston displacement for the slider crank chain shown in Fig. 50. EA is plotted vertically upwards of magnitude proportional to $\omega^2 r\left(1 + \frac{r}{l}\right)$ and FB is plotted vertically downwards of magnitude proportional to $\omega^2 r\left(-1 + \frac{r}{l}\right)$.

Calculating intermediate values and plotting these, the curve ACB is obtained. The point of zero acceleration, which corresponds to maximum velocity of the piston, is at C. It

will be noticed that the acceleration is greater in magnitude at E, corresponding to the inner dead centre position of the crank, than at F, which corresponds to the outer dead centre position. This difference is due to the obliquity of the connecting-rod.

When the ratio of $\frac{l}{r}$ is infinite the piston has simple harmonic motion, and the acceleration at the two ends of the stroke is the same in each case and of magnitude $\omega^2 r$, but of opposite sign. This is shown in Fig. 51 by the dotted line GH.

§ 69. Crank Angle for Zero Acceleration of Piston. According to the method given in § 43 for finding the velocity of the piston, the velocity is a maximum when A_1D, Fig. 16 (a), reaches its maximum value. This occurs when the crank and connecting-rod are approximately at right angles to each other. The difference between the crank angle when the crank and connecting-rod are at right angles to each other and the crank angle for zero acceleration is very small, and for a ratio of connecting-rod to crank of 4, this difference is about $\frac{3}{4}°$. For larger ratios the difference becomes less.

EXERCISES. V

1. The ratio of connecting-rod to crank in an ordinary steam engine having a stroke of 36 in. is 4·5 to 1. Determine the maximum acceleration to which the piston is subjected when the crank speed is uniform at 80 revs. per min. Also find the velocity and acceleration of the piston when the engine is running at this speed, the crank is inclined 30° to the axis of the cylinder, and the angle between the connecting-rod and crank is obtuse. [*I. Mech. E.*]

2. In an ordinary steam-engine mechanism the stroke of the piston is one-half the length of the connecting-rod. Assuming the crank-shaft to turn uniformly, draw a diagram to give the velocity of the piston at any instant. [*I. Mech. E.*]

3. Obtain an approximate expression for the acceleration of the piston of an ordinary steam engine having crank radius r and connecting-rod of length l.

The stroke of a steam engine is 28 in. and the connecting-rod is 4 ft. long. Plot the values of piston acceleration on a stroke-base when the speed of the engine is 180 revs. per min. [*I. Mech. E.*]

4. The stroke of a steam engine is 28 in. and the connecting-rod is 56 in. long. Plot a curve showing the velocity of the piston as a function of piston displacement when running at a speed of 180 revs. per min. Determine the maximum acceleration of the piston, and find the point in the stroke at which there is no acceleration of the piston. [*I. Mech. E.*]

5. Show how the configuration of a steam-engine mechanism may be determined in which the acceleration of the reciprocating parts is zero.

6. Give a geometrical construction for the determination of the acceleration of the reciprocating parts of a steam-engine mechanism for any position of the crank, and prove that your construction is correct.

7. Obtain approximate expressions for the displacement, velocity, and acceleration of the reciprocating parts of an engine in terms of θ, the crank angle displacement; l, the length of the connecting-rod; r, the radius of the crank; and ω, the number of radians per sec.

8. Show that when plotted on a piston displacement base the diagram of piston velocity for a reciprocating engine is a semicircle when the connecting-rod is infinitely long.

Show how this diagram is modified for an engine with a connecting-rod of finite length.

9. Find by graphical construction the acceleration of the piston for an engine where the crank radius is 1 ft., the connecting-rod 3 ft. 6 in. long, and the crank is at 45° from the inner dead centre. Speed—300 revs. per min.

10. A gas engine, running at 200 revs. per min., has a stroke of 12 in. and a connecting-rod 3 ft. long. Find the acceleration of the piston when the crank angle, measured from the inner dead centre, is 45°: (*a*) by Klein's construction; (*b*) by Ritterhaus's construction; (*c*) by Bennett's construction; (*d*) by calculation.

11. Repeat Question 10 for the position when the crank angle, measured from the inner dead centre, is 120°.

12. The crank of a steam-engine mechanism rotates at 120 revs. per min. The crank is 18 in. long and the connecting-rod 4 ft.

Find the acceleration of the piston for crank angles of 0, 10, 45, 90, 135, 170, and 180 degrees measured from the inner dead centre.

Plot the acceleration to a base of corresponding piston displacement, and hence find the point in the stroke at which the piston acceleration is zero and the corresponding value of the crank angle.

13. Describe, without proof, a construction for determining the acceleration of the slider in the slider crank mechanism. Apply the construction to find the acceleration of the piston of an ordinary direct acting engine when the crank is 30° from the inner dead centre. Length of crank, 8 in. Length of connecting-rod, 36 in. Speed of crank-shaft, 200 revs. per min. [*Lond. B.Sc.*]

14. Prove Klein's construction for the determination of the acceleration of any point on the connecting-rod of a steam engine. Apply it to find the acceleration of the piston of an engine—length of stroke, 21 in.; length of connecting-rod, 70 in.; revs. per min., 320, when the crank is at 30° from the 'in' dead centre. Compare your result with that obtained by treating the motion of the piston as simple harmonic. [*Lond. B.Sc.*]

15. A single-cylinder engine, running at 120 revs. per min., has a stroke of 16 in. and a connecting-rod 3 ft. long. The eccentric leads the crank by an angle of 120° and the slide valve travel is 5 in. The reciprocating parts connected to the piston rod and valve rod weigh 800 and 250 lb. respectively. Calculate the energy in these reciprocating parts when the crank displacement from the inner dead centre is 30°. Assume the valve has harmonic motion. [*Lond. B.Sc.*]

CHAPTER VI

TOOTHED GEAR WHEELS AND GEARING

§ **70. Classification of Gears.** Rotary motion between two shafts may be accomplished by means of toothed wheels or gears. Other methods of transferring motion from one shaft to another are by means of belt, rope, and chain drives. These latter methods are generally used when the distance between the centres of the shafts is large compared with the distance between two shafts connected by gearing. In many cases, where belt or rope drives are used, gear wheels can be used, but in other cases the use of gear wheels is not a practical proposition on account of the size and cost of the wheels.

Two shafts, in which motion is transmitted from one to the other, have their axes either in one plane or in different planes. The type of gearing to be used for connecting two shafts depends upon the inclination of the shafts and whether they are in one plane or in different planes. There are three cases to be considered:

(1) shafts whose axes are parallel to each other;
(2) shafts whose axes are inclined and are co-planar;
(3) shafts whose axes are inclined and which are not co-planar.

The class of gear used in each of the three cases may be roughly summarized as (1) Spur Gears, (2) Bevel Gears, (3) Spiral Gears. Worm Gears may be included in spiral gears as they are special forms of spiral gears.

§ **71. Spur Gears.** Ordinary spur gears have teeth parallel to the axis of the gear wheel. Helical gear wheels are spur wheels in which the teeth are inclined to the axis; the inclination may be either right- or left-hand. Double helical wheels may be regarded as two helical wheels back to back with right- and left-hand teeth respectively. The two inclinations very often meet at a common apex, but not necessarily so, as in the case when the teeth are *stepped.*

§ 72. **Bevel Gears.** Bevel gears are portions of cones fitted with teeth and recesses, and ordinary bevel gears have straight teeth which vary in cross-section throughout their length. Bevel gears may have their teeth inclined at an angle to the face of the bevel, in which case they become helical bevel gears or spiral bevels. The teeth on bevel wheels may be double helical, as in the case of spur gearing.

§ 73. **Spiral Gears.** The teeth of spiral gears are of necessity cut at an angle to the axis of the gear. Spiral gears, as already explained, are used to connect two shafts whose axes are inclined and which do not meet, and since the teeth of both spiral and helical gears are inclined to the axis of the gear, a helical gear may be regarded as a special case of a spiral gear. The two gears have many characteristics in common and as will be seen later the calculations are the same in both cases.

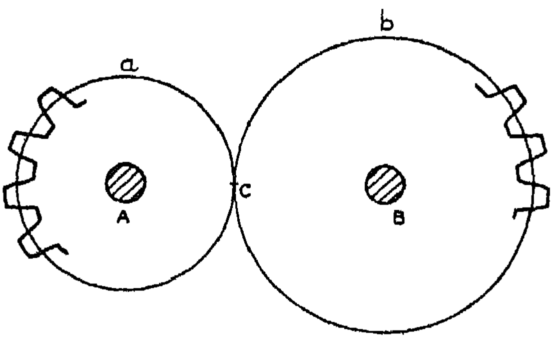


Fig. 52.

§ 74. **Rolling Motion.** To keep the problem in its simplest form, the motion between two shafts whose axes are parallel will now be considered. In Fig. 52, let A and B represent two shafts whose axes are parallel. If the power to be transmitted between the shafts is small, motion may be transferred from A to B and vice versa by plain disks represented in elevation by a and b. Assuming the disks to press against each other at the line of contact, represented by c, and that there is no slip of one periphery over the other, a motion of

one disk a, say, will result in a definite motion of the disk b. The two disks are said to roll over each other and may be termed friction wheels, since the motion between the two is due entirely to the friction between the two wheels. As the power to be transmitted increases, slip is likely to occur, and the motion between the two wheels is no longer definite. Slip may be prevented by the formation of projections and recesses on the two wheels, the projections of one wheel meshing or gearing with the recesses of the other wheel. The combination of projections and recesses results in the formation of teeth, a few of which are shown on a and b.

The motion transmitted between two toothed wheels must, for accurate work, be identical with that transmitted by two corresponding friction wheels when no slip occurs. The representation of two wheels in gear may be accomplished by means of two circles representing corresponding friction wheels, the formation of the teeth being understood. Two gear wheels connecting the shafts A and B can therefore be conveniently represented by two circles such as a and b.

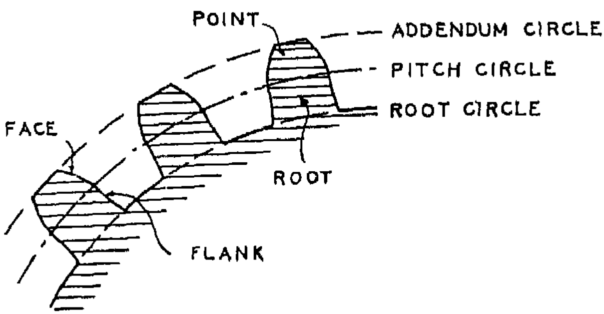


Fig. 53.

§ 75. **Definitions.** Considering a section of a spur wheel with straight teeth, Fig. 53, the addendum circle is the circle touching the extreme tips of the teeth and is the largest possible circle on the wheel. The root circle is that which passes through the base of the teeth. The intermediate circle, which corresponds to the analogous friction wheel, is the pitch circle. In Fig. 52 the circles a and b may be regarded

as pitch circles. In actual wheels the pitch circles are not readily visible, but are imaginary in the sense that there are no definite circles imprinted on the wheels, although the pitch circles exist in reality. A pitch circle may be defined as an imaginary circle representing a friction wheel which will transmit the same motion as the gear wheel.

That part of the tooth above the pitch circle is called the point, and that below the pitch circle, the root.

The surface of a tooth above the pitch circle is called the face, and that below the pitch circle, the flank. The face and flank possess depth perpendicular to the paper and each represents an area of surface.

§ 76. **Pitch.** Two gear wheels, gearing together, must have the same pitch. Unfortunately there are three different methods of measuring pitch, and these will be discussed in detail. The three types of pitch referred to are (1) circular, (2) diametral, and (3) module.

The circular pitch is the distance, measured along the pitch circle, between corresponding points on adjacent teeth. This definition, while easy to visualize, is inconvenient to apply in practice. The circular pitch is the original method of measuring pitch, and as such a large number of tools, cutters, etc., are in existence, it is impossible, on account of the cost involved, to scrap all these in favour of a more convenient system, although the more favourable systems are gradually coming into more general use.

Let p = circular pitch,

T = number of teeth on a gear wheel,

d = diameter of pitch circle.

Then, circumference of pitch circle $= \pi . d = T . p$.

$$\therefore \ p = \frac{\pi d}{T}. \qquad (1)$$

It is seen that the expression for p involves the use of π, which is an indeterminate number, and this renders calculation somewhat laborious.

The diametral pitch is the number of teeth per inch of pitch circle diameter.

Let s = diametral pitch;

then $$s = \frac{T}{d}. \qquad (2)$$

The value of s involves only the number of teeth and the pitch circle diameter, and in calculations the use of π is avoided. It must not be supposed that the use of π is entirely dispensed with, but, as will be seen shortly, it is automatically incorporated in s.

From (1) $\frac{T}{d} = \frac{\pi}{p}$; $\therefore \frac{\pi}{p} = s$ or $\pi = s.p$

The module is the number of inches of pitch circle diameter per tooth and is the reciprocal of the diametral pitch.

Let m = module pitch;

then $$m = \frac{d}{T} = \frac{1}{s}. \qquad (3)$$

One of the chief advantages of the module pitch is that the module is used in proportioning the teeth, and tooth dimensions follow much more readily than with the use of the circular pitch. The module may be in British units or metric units, the respective units being in inches and millimetres. The metric module is very much used on the Continent, where dimensions are chiefly in millimetres.

§ 77. **Tooth Proportions.** When working on the circular pitch system, the height of the tooth above the pitch circle, or addendum, is usually made $0{\cdot}3p$. The depth of the tooth below the pitch circle, or dedendum, is made $0{\cdot}4p$; the difference between the addendum and dedendum is the radial clearance.

In the diametral pitch system the addendum is made equal to $\frac{1}{s}$ and in the module system equal to m. The addendum in the diametral pitch system is thus equal to the addendum in the module system for equal pitches, since $m = \frac{1}{s}$: and these two systems are somewhat similar in use. The addendum in these two systems is slightly greater than that in the circular pitch system.

Circular pitch system, addendum $= 0 \cdot 3p$.

Diametral pitch system, addendum $= \frac{1}{s} = \frac{p}{\pi} = 0 \cdot 318p$.

Module pitch system, addendum $= m = \frac{p}{\pi} = 0 \cdot 318p$.

In the diametral and module pitch systems the dedendum is usually made equal to the addendum plus one-tenth of the thickness of tooth.

The thickness of tooth measured along the pitch line for cast gears is made equal to $0 \cdot 48p$ and for machine-cut gears is made half the circular pitch less 0·001 in. per in. of circular pitch. The width of the space along the pitch line is the circular pitch minus the width of the tooth along the pitch line. The difference between the thickness of a tooth and the corresponding space is termed backlash.

§ 78. **Blank Diameter.** The blank diameter is equal to the diameter of the addendum circle and is the diameter to which a wheel is turned before teeth are formed. The calculation of the blank diameter is readily accomplished when the pitch and number of teeth have been decided upon.

Blank diameter = pitch diameter + 2 × addendum.

Circular pitch, blank diameter $= \frac{pT}{\pi} + 0 \cdot 6p$.

Diametral pitch, blank diameter $= \frac{T}{s} + \frac{2}{s} = \frac{1}{s}(T+2)$.

Module pitch, blank diameter $= Tm + 2m = m(T+2)$.

In choosing pitches, one of the standard values must be chosen to facilitate production. As an example, a diametral pitch of 3·11 is not a practical proposition, since cutters, tools, etc., for the cutting of such a gear would not be in existence. In practice the nearest standard pitch would be chosen, i.e. 3.

The diametral and module systems being somewhat similar, it would appear that there is no particular advantage in the use of one or the other of these systems. Any advantage lies with the module pitch, as the pitch diameter calculated on this system is more convenient than that calculated on the diametral pitch system. Moreover, the distance between the

centres of two wheels in gear depends upon the two pitch diameters, and if these pitch diameters are in inches and finite decimals, workshop procedure becomes easier than is the case when the decimals are not finite.

Taking the case of a gear wheel of 50 teeth and a module pitch of 0·3, the pitch diameter is $0{\cdot}3 \times 50$ or 15 in. and the blank diameter $0{\cdot}3(50+2)$ or 15·6 in. The diametral pitch equal to a module pitch of 0·3 is $\frac{1}{0{\cdot}3}$ or 3·333..., and the nearest standard pitch to this is $3\frac{1}{2}$. For 50 teeth, pitch diameter $= \frac{1}{3\frac{1}{2}} \times 50$ or 14·2857 in., and blank diameter

$$= \frac{1}{3\frac{1}{2}}(50+2) = 14{\cdot}8571 \text{ in.}$$

This difference between the diametral and module pitch systems is further emphasized in the following table, in which values of pitch and outside diameters are calculated for different values of standard pitches, both diametral and module, for a wheel of 30 teeth.

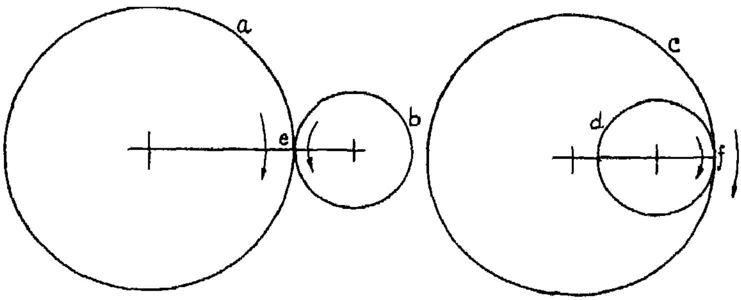
Module Pitch System			*Diametral Pitch System*		
Pitch	*Pitch diameter*	*Blank diameter*	*Pitch*	*Pitch diameter*	*Blank diameter*
0·05	1·5	1·6	20	1·5	1·6
0·1	3·0	3·2	10	3·0	3·2
0·2	6·0	6·4	5	6·0	6·4
0·25	7·5	8·0	4	7·5	8·0
0·3	9·0	9·6	$3\frac{1}{2}$	8·5714	9·1428
0·35	10·5	11·2	3	10·0	10·6667
0·4	12·0	12·8	$2\frac{1}{2}$	12·0	12·8
0·45	13·5	14·4	$2\frac{1}{4}$	13·3333	14·2222
0·6	18·0	19·2	$1\frac{3}{4}$	17·1428	18·2857
0·7	21·0	22·4	$1\frac{1}{2}$	20·0	21·3333
0·8	24·0	25·6	$1\frac{1}{4}$	24·0	25·6
0·9	27·0	28·8	1	30·0	32·0

It is seen from the above table that the pitch and blank diameters for the module system are, in general, more convenient than those for the diametral pitch system.

§ 79. **Pinion and Wheel in Gear.** When two wheels gear with each other a definite motion is transmitted from one shaft to the other. If the wheels are of equal size their

respective speeds are equal. If the wheels are of unequal size the smaller wheel is usually referred to as a pinion, the larger one as the wheel. When the larger wheel is of infinite radius, a portion of the periphery is straight and the resulting gear is termed a rack. A rack and pinion gearing together enables rotary motion to be converted into linear motion, and vice versa.

Ordinary spur gears may gear either internally or externally. Internal gear wheels have their teeth projecting inwards and are often referred to as annular wheels. External gear wheels have their teeth projecting outwards.



(a) Fig. 54. (b)

§ **80. Velocity Ratio of Spur Wheel and Pinion.** In Fig. 54 (*a*), let *a* and *b* represent the pitch circles of a wheel and pinion gearing externally. The point of contact, *e*, of the two pitch circles is on the line of centres and is termed the pitch point. Since the pitch circles roll over each other without slipping, they must rotate in opposite directions as shown by the arrows.

A wheel and pinion gearing internally is shown in Fig. 54 (*b*) at *c* and *d*. The pitch point *f* is on the line of centres produced and the two wheels rotate in the same direction.

In both cases let

r_1 = radius of larger wheel,

r_2 = radius of pinion,

ω_1 = angular velocity of larger wheel,

ω_2 = angular velocity of pinion,

N_1 = revolutions per minute of larger wheel,
N_2 = revolutions per minute of pinion,
T_1 = number of teeth on larger wheel,
T_2 = number of teeth on pinion.

Since the pitch circles roll over each other without slipping, their peripheral speeds must be equal;

$$\therefore \ \omega_1 r_1 = \omega_2 r_2.$$

Since two wheels in gear must have the same pitch,

$$\frac{r_1}{r_2} = \frac{T_1}{T_2}.$$

$$\therefore \ \frac{\omega_1}{\omega_2} = \frac{N_1}{N_2} = \frac{r_2}{r_1} = \frac{T_2}{T_1}.$$

In the case of a rack and pinion $\omega_1 r_1$ = linear speed of rack.

For two wheels in gear, if suffix 1 refers to the driving-wheel (in this case the larger wheel), the ratio $\frac{\omega_2}{\omega_1}$ is the velocity ratio or gear ratio.

§ 81. **Calculations for Spur Wheel and Pinion Gearing externally.** If the distance apart between the centres of a wheel and pinion, gearing externally, is fixed, and the velocity ratio is fixed, this automatically determines the pitch to be used. It will be a pure coincidence if the pitch determined under these conditions is a standard pitch, and in practice, in order that a standard pitch may be used, it is necessary to make a compromise between the distance apart of the wheel centres and the velocity ratio.

Example 1. Find the numbers of teeth on a wheel and pinion to give a velocity ratio of 3 to 1, the distance between the centres being approximately $10\frac{3}{8}$ in.

Let c = distance apart of centres,

then $$d_1+d_2 = 2c = 20{\cdot}75 \text{ approx.};$$

also $$d_1 = 3d_2.$$

$$\therefore \ 3d_2+d_2 = 20{\cdot}75 \text{ or } d_2 = 5{\cdot}1875 \text{ in. approx.}$$

and $$d_1 = 15{\cdot}5625 \text{ in. approx.}$$

The numbers of teeth in the two wheels must, of course, be whole numbers, and to ensure the teeth having maximum strength the number of teeth on the pinion should be as small as possible.

In practice, 12 is the least number of teeth allowed under ordinary conditions.

Assuming $T_2 = 12$, then a possible value of m is $\dfrac{5{\cdot}1875}{12}$ or 0·4323. This is an odd pitch, the nearest standard pitch being 0·45. Using this value of m and $T_2 = 12$, the exact value of d_2 is

$$12 \times 0{\cdot}45 = 5{\cdot}4 \text{ in.} \quad \therefore\ d_1 = 3 \times 5{\cdot}4 = 16{\cdot}2 \text{ in.}$$

$$\therefore \text{ distance apart of centres} = \frac{16{\cdot}2 + 5{\cdot}4}{2} = 10{\cdot}8 \text{ in.}$$

The value thus found varies by an appreciable amount from the specified distance, viz. $10\frac{3}{8}$ in. If this variation is too great, a nearer value may be obtained by increasing the number of teeth on the pinion. This number may be obtained by dividing 5·1875 by standard pitches until a result is obtained which is very nearly a whole number.

$$\text{Thus } \frac{5{\cdot}1875}{0{\cdot}4} = 12{\cdot}97,\quad \frac{5{\cdot}1875}{0{\cdot}35} = 14{\cdot}82,\quad \frac{5{\cdot}1875}{0{\cdot}3} = 17{\cdot}29.$$

Of these values, a standard module pitch of 0·4 gives a result very near to 13. Using the value of $T_2 = 13$ and $m = 0{\cdot}4$,

$$d_2 = 5{\cdot}2 \text{ in. and } d_1 = 15{\cdot}6 \text{ in.,}$$

whence
$$c = \frac{15{\cdot}6 + 5{\cdot}2}{2} = 10{\cdot}4 \text{ in.}$$

§ **82. Condition that Shapes of Wheel Teeth must satisfy.** It is a well-known proposition in geometry that two curves in contact at a point must have a common normal at that point. One essential condition to be satisfied by two teeth in contact is that the common normal to the teeth in contact must pass through the pitch point, this being the common point of contact between the two pitch circles.

In Fig. 55, let A and B be the centres of two wheels in gear, a and b the respective pitch circles, and O the pitch point. Let E be the point of contact of two teeth in contact and $DOEC$ the common normal, assumed for the moment to pass through the pitch point. The lines AD and BC are drawn perpendicular to the common normal. Further conditions to be satisfied are that the motion between the wheels is equivalent to that of one pitch circle rolling over the other and that the velocity ratio is constant.

The triangles ADO and BCO are similar;

$$\therefore \frac{AD}{AO} = \frac{BC}{BO} \quad \text{or} \quad \frac{AD}{BC} = \frac{AO}{BO}.$$

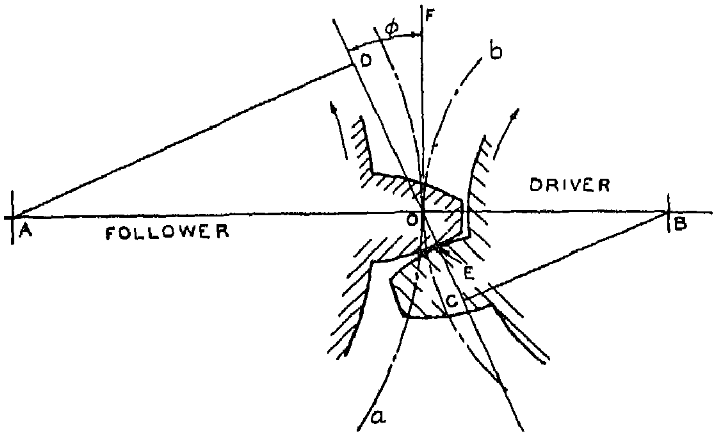


Fig. 55.

The ratio $\frac{AO}{BO}$ is the ratio of the radii of the two pitch circles and is equal to the velocity ratio, which is constant: $\therefore \frac{AD}{BC}$ is equal to the velocity ratio and this ratio of $\frac{AD}{BC}$ can only be constant if DC passes through O. If DC does not pass through O, the ratio of $\frac{AD}{BC}$ is no longer constant and the wheels do not satisfy the condition that the velocity ratio shall be constant.

In Fig. 55 the common normal $DOEC$ is drawn through the pitch point O. A more rigid proof that the common normal must pass through the pitch point may be developed by considering the case when the common normal is not assumed to pass through the pitch point. In Fig. 56 let A and B represent the centres of two wheels in gear, and O the pitch point. Let E represent the point of contact of two teeth in gear and DO_1EC the common normal cutting the line of centres AB at O_1. Draw AD and BC perpendicular to DC. Join AE and BE and draw EF perpendicular to AE and EG perpendicular to BE.

Velocity of E on wheel A is along EF and its magnitude is $\omega_A . AE$. Similarly, velocity of E on wheel B is along EG and its magnitude is $\omega_B . BE$. The components of these velocities along the common normal must be equal since

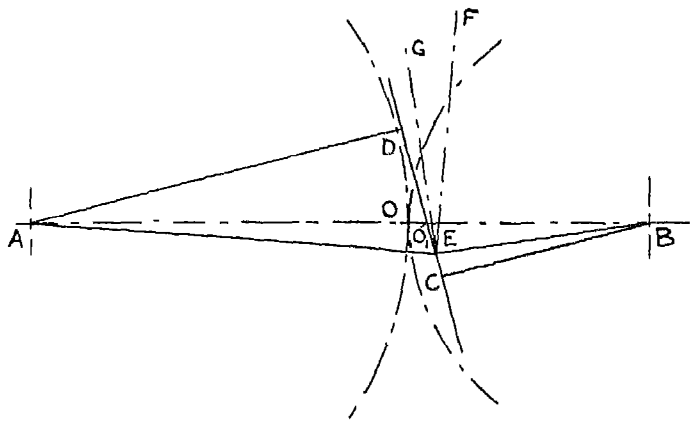


Fig. 56.

penetration or separation of teeth cannot occur; the only possible relative motion between the teeth at the point of contact is that of sliding, i.e. perpendicular to the common normal. Since the components of these velocities of E along the common normal are equal, then

$$\omega_A . AE \cos FED = \omega_B . BE \cos GED,$$

$$\omega_A . AE \sin DEA = \omega_B . BE \sin DEB,$$

$$\omega_A . AD = \omega_B . BC,$$

i.e.
$$\frac{\omega_B}{\omega_A} = \frac{AD}{BC}.$$

Since the triangles ADO_1 and BCO_1 are similar,

$$\frac{AD}{BC} = \frac{AO_1}{BO_1}.$$

For constant velocity ratio $\frac{\omega_B}{\omega_A}$ must be equal to $\frac{AO}{BO}$ and consequently

$$\frac{\omega_B}{\omega_A} = \frac{AD}{BC} = \frac{AO_1}{BO_1} = \frac{AO}{BO}.$$

Thus $\frac{AO_1}{BO_1}$ must be equal to $\frac{AO}{BO}$ and this can only be true if O and O_1 coincide, in which case the common normal passes through the pitch point O.

The angle between the common normal and the common tangent to the two pitch circles is the angle of obliquity. This angle is usually designated by ϕ and in Fig. 55 is the angle DOF.

§ **83. Profiles used in Practice.** For two wheels gearing together, the profile of a tooth on one wheel is arbitrary, providing the profile of the mating tooth is made to satisfy the condition outlined in § 82. For manufacturing purposes the profiles must be of some form which is capable of reproduction on machines. In practice it is found that two geometrical curves, the cycloid and the involute, fulfil the necessary conditions.

§ **84. Cycloidal Curves.** A cycloid is the path traced out by a point on a circle which rolls without slipping over a straight line. Profiles generated in this manner are used for rack teeth. For wheel teeth the profile used is that obtained when generating circles roll inside and outside the pitch circle of a particular wheel. These curves are called hypocycloid and epicycloid respectively. In Fig. 57, let ABC represent the pitch circle of a wheel whose centre is at O. A rolling circle a, rolling on the inside of the pitch circle, will generate the hypocycloid AEB, a small portion of which, near the pitch circle, is used for the profile of the flank of the tooth. The tracing point E, on the rolling circle, coincides with A when the rolling circle is in its initial position and with B when in its final position. Since the circle rolls without slipping, the arc AD on the pitch circle is equal to the arc ED on the rolling circle, and it is a property of the cycloid that the line ED is normal to the curve; this normal, therefore, always passes through the pitch point, if D is regarded as the pitch point.

The face of the tooth may be generated by a rolling circle of different diameter, such as b, rolling on the outside of the pitch circle. The epicycloid generated in this manner is shown in Fig. 57 by BFC. Commencing at B, the tracing

point F coincides with B when the rolling circle is in its original position and with C when in its final position. Again the arc BG = arc GF and FG is normal to the epicycloid.

It will be noticed that the contour of the tooth changes at the pitch circle and for this reason cycloidal teeth are often termed *double curve* teeth. It might be mentioned here that

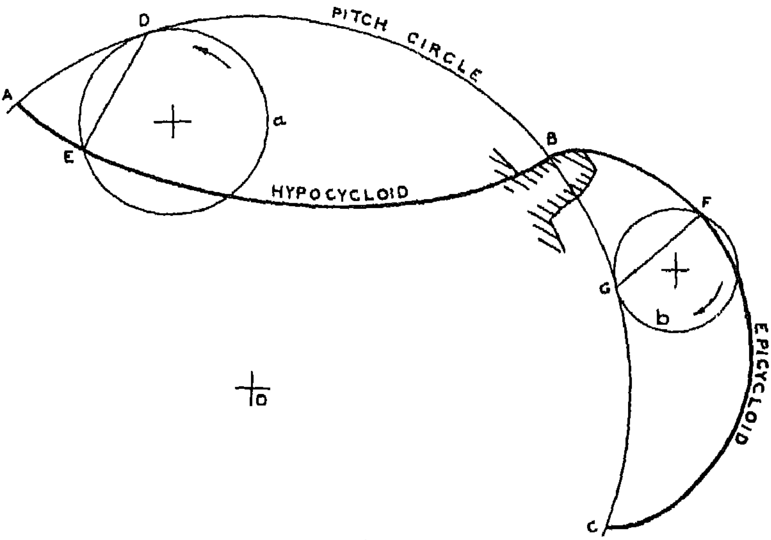


Fig. 57.

cycloidal teeth are rarely used in modern practice, particularly in machine-cut gears, but they are sometimes met with in cast gears.

For two cycloidal gears to gear correctly, the face of a tooth on one wheel must be generated by the same size of rolling circle as that which generates the flank of the mating tooth. For a number of gears to be interchangeable the same size of rolling circle must be used throughout for generating the faces and flanks.

§ **85. Involute Curve.** When a straight line rolls without slipping over the circumference of a circle, a point on the straight line describes an involute. Alternatively, an involute is the path traced out by a point on the extremity of a taut string which is unwound from a circle. In Fig. 58, let a repre-

sent the pitch circle of a wheel and AC a circle of lesser diameter called the base circle. Let BC represent the position of a line rolling over the circumference of the base circle. The tracing point B in its original position coincides with A, moves through the point B, and, on further rolling, through D. The path ABD is an involute and a small portion of the

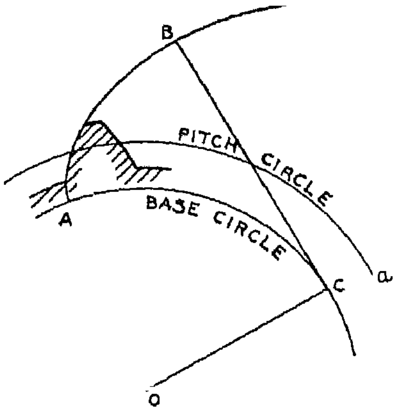


Fig. 58.

involute near the base circle is used for the tooth profile. The arc AC = length of string unwound BC, and it is a property of the involute that BC is normal to the curve, since, for the given figuration, C may be regarded as the instantaneous centre of rotation of B, and the motion of B must be perpendicular to BC. The profile of the tooth is a single curve, and involute teeth are often termed *single curve* teeth.

The use of involute teeth has almost entirely superseded that of cycloidal shape, owing to the ease of standardization, manufacture, cost of production, etc. In modern practice teeth are automatically generated, and one cutter or tool suffices for a complete set of interchangeable gears. All gears will gear with their corresponding rack, and hence by making the cutter in the form of a rack, this cutter will generate, when used in suitable generating machines, correct gears for any conveniently sized wheel. Further, the teeth of an involute rack are straight, and cutters of this form can be made to a high degree of accuracy.

§ **86. Angle of Obliquity.** In Fig. 59, let a and b be the pitch circles of two gears whose centres are at A and B respectively. Let AD and BC be the radii of the base circles, these being drawn such that the common tangent CD passes through the pitch point O. Then the profile of a tooth on a in contact with the profile of a tooth on b will have the

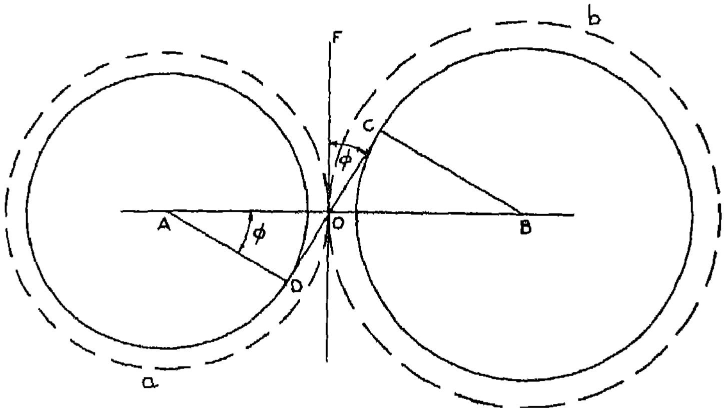


Fig. 59.

common point of contact on CD, since the normal to the two curves must pass through the pitch point. In other words the common normal to two teeth in contact is tangential to both base circles and is constant in direction, giving a constant angle of obliquity.

This property is of great practical importance since it means that the direction of pressure between teeth is always constant. For this reason the angle of obliquity is often called the *pressure angle*. The greater the relative difference between the base and pitch circles, the greater is the pressure angle. The angle of obliquity or pressure angle adopted by many firms in this country is $14\frac{1}{2}°$, though it has been found advantageous in some cases to increase this to as much as $22\frac{1}{2}°$. In practice, then, the angle of obliquity is rarely less than $14\frac{1}{2}°$ and seldom greater than $22\frac{1}{2}°$. When the pressure angle exceeds $22\frac{1}{2}°$ the loss due to friction becomes excessive. Generally speaking, the smaller angle should not be used for gears of approximately equal size.

§ 87. **Diameter of Base Circle.** When the pressure angle is decided upon, the diameter of the base circle readily follows. In Fig. 59 AD is the radius of the base circle for wheel a, and $AD = OA \cos\phi$.

Diameter of base circle = pitch diameter × cos pressure angle.

For a pressure angle of $14\frac{1}{2}°$, $\cos 14\frac{1}{2}° = 0{\cdot}96815$.

„ „ „ $22\frac{1}{2}°$, $\cos 22\frac{1}{2}° = 0{\cdot}92388$.

The pressure angle recommended by the British Standards Institution is 20°.

§ 88. **Variation of Distance between Wheel Centres.** Since the common normal to two teeth in contact passes through the pitch point and is tangential to both base circles, the distance between the centres of the wheels may be increased slightly without affecting the transmission of uniform motion. Increasing the distance between the centres automatically increases the pressure angle, though but slightly, but the line of pressure or line of action still passes through the pitch point. This important characteristic is peculiar to involute teeth and is not possessed by cycloidal teeth.

§ 89. **Internal Involute Gears.** A pinion gearing with an internal gear rotates in the same direction. With internal gears the pinion cannot of course be made equal in size to the wheel, and the best results are obtained when there is a considerable difference between the pitch diameters of wheel and pinion. In special cases, however, pinions have been made in which the difference is small—a difference of perhaps three or more teeth.

It might at first be thought impossible to construct an involute inside a wheel, but if the shape of the tooth space of an internal gear is made identical with the tooth profile of an external gear of an equal number of teeth, this difficulty is overcome.

§ 90. **Length of Line of Action.** The length of the line of action is readily determined by finding the point of intersection of the pressure line, or line of action, with the addendum circle.

§ 91. **Involute Rack.** The teeth of an involute rack have

straight faces whose normals are inclined at the required pressure angle to the pitch line of the rack. The faces of the teeth are inclined at the same angle to the base of the rack as shown in Fig. 60, in which the pitch line is represented by AB.

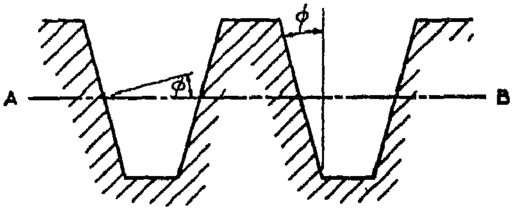


Fig. 60.

§ 92. Choice of Rolling Circle for Cycloidal Teeth. When generating the flank of a cycloidal tooth, the size of the rolling circle, rolling inside the pitch circle, determines the shape of the tooth at the root; and since the shape at the root determines the strength of the tooth to resist bending action,

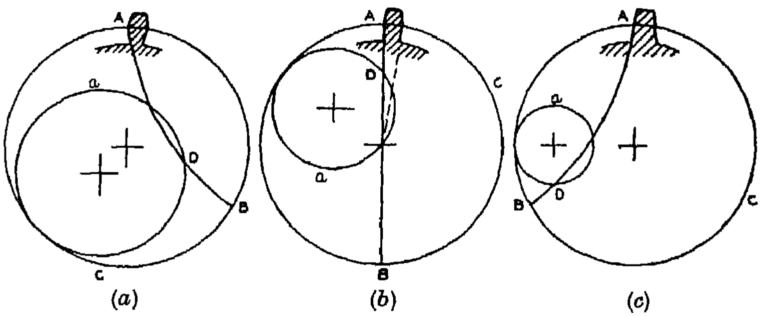


Fig. 61.

the choice of size of rolling circle is an important matter. In Fig. 61 (a), ABC represents the pitch circle for a gear wheel and a the rolling circle to generate the flank. In this case the diameter of the rolling circle is greater than the radius of the pitch circle, and the resulting hypocycloid is ADB. It will be seen that the tooth profile narrows from the pitch line to the root, giving a weak form of tooth.

In (b) the diameter of the rolling circle a is equal to the radius of the pitch circle, and the resulting hypocycloid is a diameter ADB. The flank, from the pitch circle to the root,

is a straight line which is radial, and the tooth is said to have radial flanks.

In (*c*) the rolling circle *a* is less than the radius of the pitch circle, and part of the hypocycloid *ADB* is used for the flank, from which it is seen that this gives the strongest form of tooth. In general, the diameter of the rolling circle must be kept small compared with the pitch circle. In practice, the rolling circle cannot be made too small on account of the increase in the maximum angle of obliquity or pressure angle. For a set of interchangeable gears it is usual to make the diameter of the rolling circle equal to the radius of the smallest pitch circle.

§ 93. **Path of Contact of Two Cycloidal Teeth.** Referring to Fig. 57, the epicycloid *BFC* is the path traced out by a point *F* on the rolling circle *b*, rolling on the outside of the pitch circle *ABC*; hence the point of contact, on the face of the tooth, with a point on the flank of the mating tooth must lie on the rolling circle. The movement of the rolling circle is relative to the pitch circle, and as the pitch circle rotates the rolling circle must move with the pitch circle, and the point of contact thus moves. Referring to Fig. 55, the common normal to the profiles of two teeth in contact passes through the pitch point, hence the common normal cuts each curve of tooth profile at its point of intersection with the rolling circle, i.e. the point of contact is along the arc of the rolling circle when a point on the rolling circle coincides with the pitch point. It follows then that the angle of obliquity is variable, having a maximum value for the position in which the rolling circle (in contact with the pitch point) cuts the addendum circle.

To illustrate these points further, consider Fig. 62, in which *bb* and *cc* are two pitch circles of which *cc* is the driver. The rolling circle *a* is shown in contact with the pitch point *O*. The circles *dd* and *ee* are the addenda circles of the pitch circles *bb* and *cc* respectively. The tracing point *P* on the rolling circle traces out the epicycloid *FP* on pitch circle *bb* and traces the hypocycloid *EP* on the pitch circle *cc*. *P* is the point of contact of the tooth profiles when two teeth are just

coming into gear. The line of pressure is along OP and the angle of obliquity or pressure angle is POQ. When E and F coincide at O, the line of pressure is along OQ, showing that the obliquity has altered from the angle POQ to zero. The path of contact is along the arc PO, since the common normal cuts each tooth profile at its intersection with the rolling circle, and the complete path of contact is POP_1.

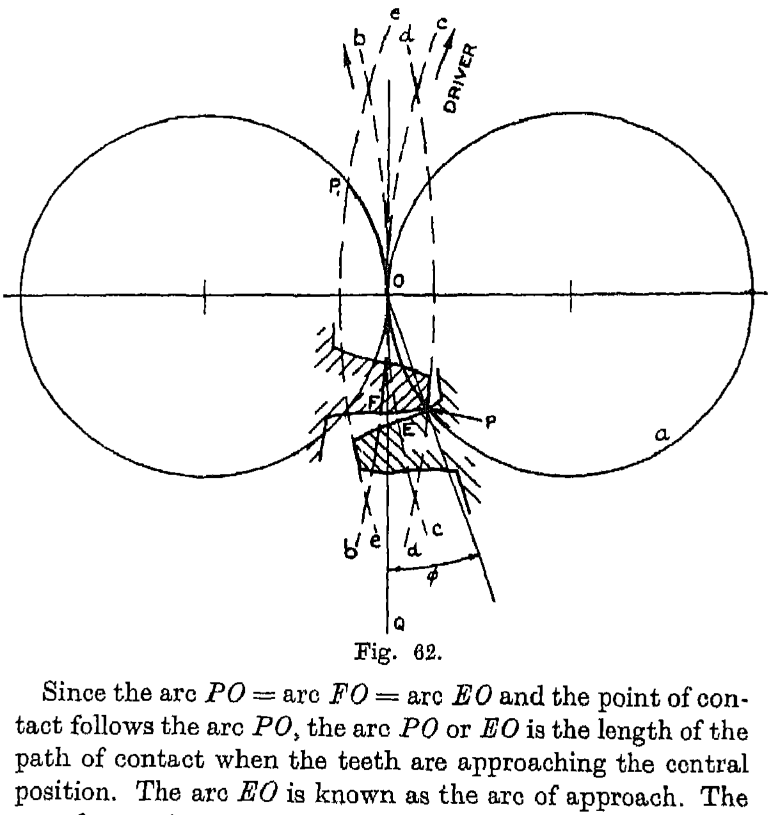


Fig. 62.

Since the arc PO = arc FO = arc EO and the point of contact follows the arc PO, the arc PO or EO is the length of the path of contact when the teeth are approaching the central position. The arc EO is known as the arc of approach. The arc of recess is a corresponding arc on bb on the opposite side of O, which is equal to OP_1. If two teeth are always to be in contact, the arc of contact (which is the arc of approach plus the arc of recess) should be twice the circular pitch.

§ **94. Velocity Ratio of Two Bevel Gears.** Bevel gears are used to gear two shafts whose axes meet and which are

in one plane. In Fig. 63, let OA and OB represent two axes in one plane meeting at O, which is known as the apex. The motion between the two axes can be transmitted by cones FOD and DOG, but only the frusta are used as shown in the figure. The formation of teeth on the cones to prevent slipping converts the frusta into bevel gears. The cones are 'backed' by extra metal to enable teeth to be cut.

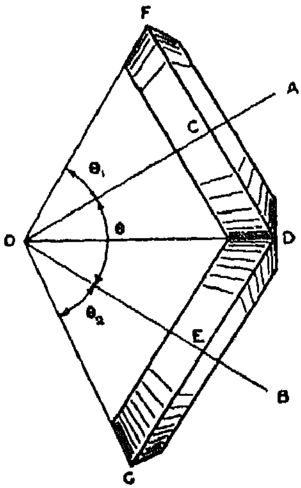


Fig. 63.

Let θ = angle between the axes of the shafts = angle AOB,
θ_1 = semi angle of cone on axis OA,
θ_2 = semi-angle of cone on axis OB,
ω_1 = angular speed of cone on axis OA,
ω_2 = angular speed of cone on axis OB
T_1 = no. of teeth on bevel on axis OA.
T_2 = no. of teeth on bevel on axis OB,

then

$\omega_1 . CD = \omega_2 . ED$, since the cones roll without slipping

$$\therefore \frac{\omega_2}{\omega_1} = \frac{CD}{ED}.$$

Now,

$$\frac{CD}{ED} = \frac{CD}{OD} \times \frac{OD}{ED} = \frac{CF}{OF} \times \frac{OG}{EG} = \sin\theta_1 \times \frac{1}{\sin\theta_2} = \frac{\sin\theta_1}{\sin\theta_2}.$$

$$\frac{\omega_2}{\omega_1} = \frac{\sin\theta_1}{\sin\theta_2} = \frac{N_2}{N_1} = \frac{T_1}{T_2}.$$

When the two shafts are at right angles to each other and rotate at equal speeds, the bevel wheels have equal numbers of teeth, and the gears are known as mitre gears.

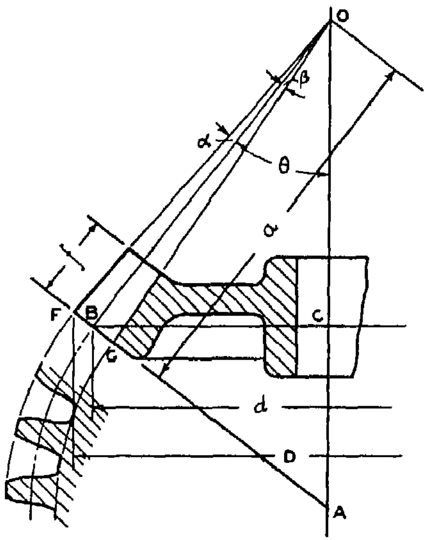


Fig. 64.

§ 95. **Definitions for Bevel Gears.** The pitch angle is the angle between the pitch cone and the axis. In Fig. 64, OA represents the axis of a bevel gear, OB is the pitch cone. The angle BOC is the pitch angle, indicated by θ.

The pitch diameter is the diameter of the largest part of the pitch cone, i.e. $d = 2BC$.

The face angle is the angle between the outside face of the bevel FO and the axis, i.e. angle FOC.

The top angle is the angle between the outside face of the bevel and the pitch cone, i.e. angle $FOB = \alpha$. The face angle is thus equal to the pitch angle plus the top angle.

The bottom angle is the angle between the pitch cone surface OB and the bottom of the teeth GO.

The apex distance a is measured from the apex to the back

edge GF. The back edge is at right angles to the pitch cone surface OB.

The outside diameter is the maximum diameter D measured to the extreme tip of the teeth at F.

From the geometry of Fig. 64, the following relations hold:

Apex distance, $$a = \frac{d}{2\sin\theta}.$$

Top angle,

$$\tan\alpha = \frac{\text{addendum}}{\text{apex distance}} = \frac{FB}{a} = \frac{m}{a},\ \text{where } m = \text{module pitch.}$$

Bottom angle, $$\tan\beta = \frac{\text{dedendum}}{\text{apex distance}}.$$

Face angle = pitch angle + top angle.

Outside diameter = pitch diameter + 2 × addendum × $\cos\theta$,

$$= d + 2m\cos\theta.$$

The proportions of bevel teeth are calculated from a normal section as seen on the back edge FG. The true shape of the teeth can be seen by developing the back edge, that is, by projecting FG to meet the axis at A. A circular arc drawn with AB as radius is the development of the pitch diameter, and the addendum and dedendum are set out above and below this circular arc respectively. In the figure, FB is equal to the addendum and BG the dedendum.

§ 96. **Helical Gearing.** Ordinary spur gears with teeth parallel to the axis are not suitable for high speeds, owing to the action of the teeth being somewhat irregular and thus causing noise and vibration. This irregular action can be remedied to a certain extent by using several gear disks fastened together in such a manner that each disk is slightly in advance of the previous one, as shown in Fig. 65. A wheel obtained in this manner is a *stepped* wheel. A single helical wheel is a development from this idea by diminishing the thickness of the disks until ultimately the resulting teeth are replaced by continuous teeth, which are inclined to the axis of the wheel. If the angle of inclination is correctly chosen together with a suitable width of face, it is possible to obtain

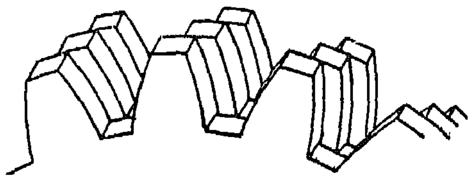


Fig. 65.

continuous pitch-line contact. This continuous pitch-line contact ensures much smoother running characteristics, as part of the load is always being transmitted at the pitch line.

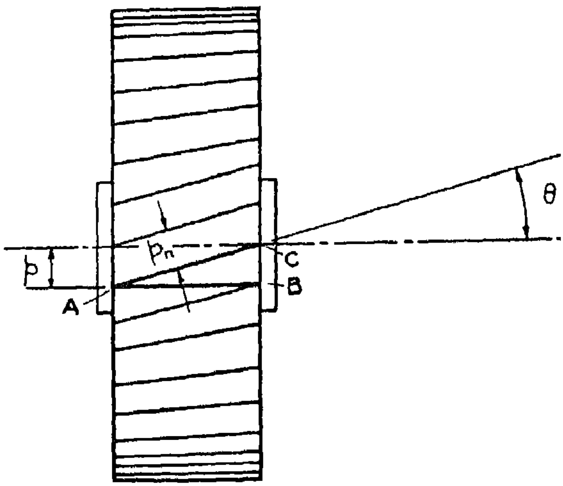


Fig. 66.

In Fig. 66 an elevation of a helical gear is shown in which the teeth are inclined to the axis at an angle θ, called the spiral angle. It will be readily seen that the true shape of a tooth is that perpendicular to the line of inclination of the tooth. When cutting helical teeth with a rotary cutter, the cutter is inclined to the axis of the wheel to suit the spiral angle, and the pitch of the cutter has to be designed to suit the normal pitch of the teeth, which is always smaller than the circular pitch. The normal pitch is the shortest distance measured along the pitch line between corresponding points on adjacent teeth.

Let $\qquad p_n =$ normal pitch,

$p =$ circular pitch;

then $\qquad p_n = p \cos\theta.$

The circular pitch p multiplied by the number of teeth is equal to the circumference of the pitch circle;

or $\qquad p \times T = \pi d.$

The normal pitch can be expressed in terms of the diametral or module pitch; thus

$$s_n = \frac{s}{\cos\theta},$$

$$m_n = m\cos\theta.$$

The addendum is measured by the normal module and in magnitude is m_n.

The minimum width of face to ensure continuity of pitch-line contact is readily found from the triangle ABC, Fig. 66, in which the point B will be just commencing pitch-line contact as A is just finishing.

$$AB\tan\theta = BC,$$

or $\qquad AB$, the minimum width $= \dfrac{\text{circular pitch}}{\text{tan spiral angle}}.$

A single helical gear of one hand must gear with one of the opposite hand.

§ 97. **Double Helical Gearing.** One of the disadvantages of single helical gearing is the provision that must be made for side thrust. The tangential pressure on the pitch line induces a pressure or side thrust parallel to the axis, and the magnitude depends upon the spiral angle. In practice this disadvantage is overcome by using two helical wheels of opposite hand side by side, gearing with corresponding helical wheels, or more conveniently by using double helical wheels in which the side thrusts balance each other. Such a gear is shown in Fig. 67. Gears of this nature have proved so successful in practice in transmitting heavy loads at high speeds, that they are very largely used on all classes of high and low speed gears. Speeds up to 10,000 ft. per min. have been used and single units of over 20,000 horse-power have been constructed.

The manufacture of double helical gears has now reached a very high degree of accuracy and the teeth can be cut by one of several methods, including milling, planing, and hobbing. Continuous teeth cannot be cut by the milling and hobbing methods, but by cutting away the centre portion where the teeth of the opposite hand meet, these methods can be successfully used. Double helical gears without a central gap can be cut by means of an end mill, but as it is of necessity small in diameter and has few cutting edges, it is subject

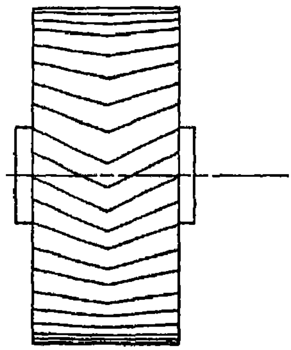


Fig. 67.

to rapid wear; more recent methods introduced for cutting 'dead end' double helical teeth are by planing by means of rack cutters or by circular cutters in the form of pinions. The provision of a central gap greatly facilitates production and is not very detrimental when the face of the gear exceeds five times the circular pitch.

§ 98. **Spiral Gearing.** Two shafts in parallel planes whose axes do not meet, but which are inclined at an angle to each other, may be connected by spiral gearing. A helical gear can be used as a spiral gear and vice versa, if the mating gear is made to suit.

Let p = circular pitch,

p_n = normal pitch,

θ = spiral angle,

T = number of teeth,

d = diameter of pitch circle.

Then, from what has been said about helical teeth,

$$pT = \pi d \quad \text{and} \quad p_n = p \cos \theta.$$

$$\therefore \frac{p_n T}{\cos \theta} = \pi d.$$

From this latter equation it is seen that the pitch diameter may be altered by altering the spiral angle without affecting

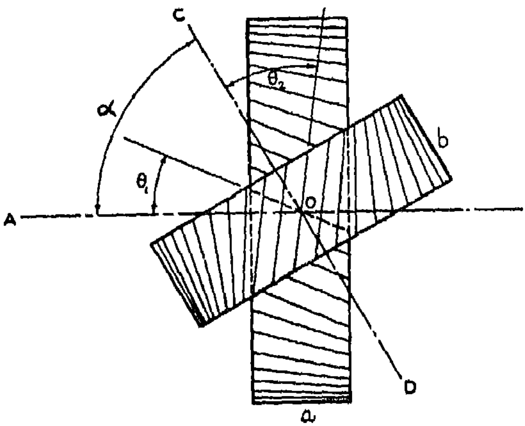


Fig. 68.

the number of teeth. For two spirals in gear the normal pitch must be the same, but the circular pitches may be, and quite often are, different. The diameters of the pitch cylinders do not affect the velocity ratio, which depends directly upon the number of teeth.

Referring to Fig. 68, let AB and CD represent two axes in parallel planes, the axes not meeting and inclined to each other. The cylinders a and b represent the pitch cylinders whose point of contact is at O. For a given velocity ratio the possible variables make the problem of determining the necessary dimensions more difficult. In general there are two methods of attack:

(1) To assume a spiral angle for each cylinder and solve for the pitch diameters.

(2) To assume the pitch diameters and solve for spiral angles.

Further, any number of solutions may be obtained by using different spiral angles, the only condition to be satisfied in this case being that if the spirals are of the same hand, the sum of the spiral angles must be equal to the angle between the two shafts. Usually the method of solution is to assume spiral angles to fulfil this condition and solve for the pitch diameters.

Let α = angle between the shafts,
p_n = normal pitch,
p = circular pitch,
d_1 = diameter of pitch cylinder a,
d_2 = diameter of pitch cylinder b,
θ_1 = spiral angle of a,
θ_2 = spiral angle of b,
l = shortest distance between shaft centres,
T_1 = number of teeth on a,
T_2 = number of teeth on b,
ω_1 = angular speed of a,
ω_2 = angular speed of b.

Then,
$$p_n = p\cos\theta,$$
and
$$\frac{p_n T_1}{\cos\theta_1} = \pi d_1, \quad \text{or} \quad p_n = \frac{\pi d_1 \cos\theta_1}{T_1}.$$

But p_n must be the same for both spirals.
$$\therefore \frac{d_1\cos\theta_1}{T_1} = \frac{d_2\cos\theta_2}{T_2};$$
also
$$\frac{\omega_1}{\omega_2} = \frac{T_2}{T_1} = \frac{d_2\cos\theta_2}{d_1\cos\theta_1},$$
and
$$\frac{d_1}{2}+\frac{d_2}{2} = l.$$

An example will perhaps make this clear.

EXAMPLE 2. Two axes are to be connected by spiral gearing. The shortest distance between the axes is 8 in., the axes are inclined at an angle of 70°, and the velocity ratio is 3. Find suitable pitch diameters.

Any number of solutions may be obtained by making
$$\theta_1+\theta_2 = 70°.$$

(1) Assume $\theta_1 = 30°, \quad \theta_2 = 40°.$

Then $$\frac{\omega_1}{\omega_2} = 3 = \frac{T_2}{T_1} = \frac{d_2 \cos\theta_2}{d_1 \cos\theta_1}.$$

$$\therefore \frac{d_2}{d_1} = 3 \times \frac{\cos 30°}{\cos 40°} = 3{\cdot}392:$$

also $$d_1 + d_2 = 16.$$

$$\therefore d_1 + 3{\cdot}392 d_1 = 16$$

$$d_1 = 3{\cdot}643 \text{ in.}$$

and $$d_2 = 16 - 3{\cdot}643 = 12{\cdot}357 \text{ in.}$$

(2) Alternative solution:

Assume $\theta_1 = 40°, \quad \theta_2 = 30°.$

Then $$\frac{\omega_1}{\omega_2} = 3 = \frac{T_2}{T_1} = \frac{d_2 \cos\theta_2}{d_1 \cos\theta_1},$$

$$\frac{d_2}{d_1} = 3 \times \frac{\cos 40°}{\cos 30°} = 2{\cdot}653,$$

$$d_1 + 2{\cdot}653 d_1 = 16.$$

$$\therefore d_1 = 4{\cdot}380 \text{ in.}$$

and $$d_2 = 11{\cdot}620 \text{ in.}$$

When two spirals are of opposite hand the difference between their spiral angles must be equal to the angle between their shafts. For two shafts at 90° both spirals must be of the same hand. In practice, spirals must be designed for standard normal pitches.

§ 99. **Worm Gearing.** A worm gear is a particular form of spiral gear, in which the larger wheel has a hollow face so as to allow the teeth to envelop a portion of the pitch diameter of the other gear. In worm gearing the smaller wheel, usually called the worm, has a large spiral anglo, and it is usual to express the inclination of the teeth as lead angle, which is complementary to the spiral angle. In most types of worm gears the worm and the worm wheel are at right angles.

The worm becomes a form of screw thread as shown in Fig. 69. The contour of a tooth on the worm is straight sided, the side being inclined at an angle corresponding to the pressure angle, which is usually 14½° for lead angles up to 20°. For greater lead angles the pressure angle is increased. Single

worms have one continuous thread, double-threaded worms have two continuous threads or two starts. Worms may be constructed with as many as ten starts. The greater the number of starts, the greater the lead angle, other conditions being equal. In rear axle transmission of motor-cars worms of four starts are commonly used. It is found in practice that the most efficient worm gears are those in which the lead angle is high, that is, from 40° to 45°. By having a multiple number of starts the worm wheel may be made less for a given velocity ratio.

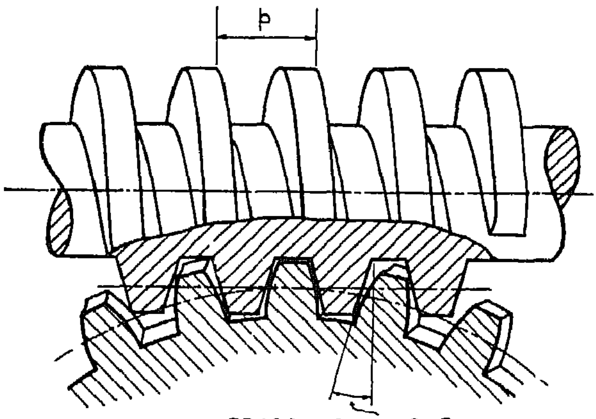


Fig. 69.

Linear pitch is the distance measured along the pitch line between corresponding points on the adjacent threads. This is shown as p in Fig. 69.

The circular pitch is the distance, measured along the pitch circle of the worm wheel, between corresponding points on adjacent teeth. For ordinary worm gears in which the axis of the worm is at right angles to the worm wheel, the circular pitch is equal to the linear pitch. The dimensions of teeth both of worm and worm wheel must be calculated from the normal pitch.

The lead angle is the angle between the thread of the worm and a line perpendicular to the axis. This angle is the complement of the spiral angle.

The lead of the worm (which must be carefully distinguished from linear pitch) is the distance measured along the pitch line between corresponding points on the same thread. Alternatively the lead is the linear pitch multiplied by the number of threads or starts.

For single-threaded worms the velocity ratio is equal to the number of teeth on the worm wheel; for double-threaded worms it is half the number of teeth.

In the manufacture of worm gearing, or indeed of any class of gearing, special precautions and refinements are made to ensure smoothness and accuracy. These points cannot very well be dealt with here, and the student is referred to the many special books on wheel teeth and gearing.

EXERCISES. VI

1. Explain why involute and cycloidal curves give suitable shapes for the profiles of the teeth of gear wheels.

2. Show that involute teeth will gear correctly when the distance between the centres of two gear wheels varies slightly.

3. Prove that the condition that the velocity ratio between two toothed wheels should be constant is that the common normal at the point of contact of the two teeth must pass through the point of contact of the two pitch circles.

Show that, when teeth of involute or cycloidal form are employed, this condition is satisfied.

4. Define the following terms: cycloidal and involute teeth (which are most used, and why ?), rolling circle, addendum, addendum circle, pitch line, arc of recess, arc of approach, line of action. Show that with involute teeth the line of action is a straight line tangent to the base circles. [*Inst. C. E.*]

5. Explain the cause of noise, vibration, or wear of wheel teeth.

6. What conditions have to be satisfied in the design of the teeth in wheel trains for screw-cutting machinery ?

Discuss the essential differences (practical) between such teeth and those used in double reduction gear for turbine engines. [*I. Mech. E.*]

7. Two parallel shafts are to be geared by toothed gearing so that the velocity ratio of the shafts is 3 to 1 and the distance between the shafts is to be as near as possible 14 in. Determine the pitch-circle diameters and the number of teeth on the spur and pinion wheels. The number of teeth on the pinion must not be less than 12.

Draw the true shape of an involute tooth on the pinion, choosing an angle of obliquity of $22\frac{1}{2}°$. [*I. Mech. E.*]

8. The pitch-circle diameter of a spur wheel is 8 in. and there are 16 involute teeth $1\frac{1}{2}$ in. wide. The angle of obliquity of thrust is 20°. Make a full-size drawing of the profile of a tooth. [*I. Mech. E.*]

9. Two parallel shafts, the axes of which are 2 in. apart, are to be connected by internal toothed gearing. The required velocity ratio is 2, the pinion is to have 12 involute teeth, and the angle of obliquity is to be 20°. Design the gearing, showing enough in your drawing to determine the true shape of the teeth. [*I. Mech. E.*]

10. Two shafts in the same plane but at right angles are to be connected by bevel wheels. The velocity ratio is to be 2. The pitch diameter of the smaller bevel is 5 in. and the number of teeth 15. Sketch these bevel wheels, and show how you would determine the shape of the involute teeth. [*I. Mech. E.*]

11. The least distance between two shafts which are inclined 60° to each other is 14 in. Design screw (spiral) gearing which will give a velocity ratio for these shafts of 3 to 1. [*I. Mech. E.*]

12. A shaft running at 100 revs. per min. drives another at 300 revs. per min. The distance between the axes of the shafts is 3 ft.

Find the diameter of the pitch circles of the gear wheels required.

13. Find the diameter of the pitch circle of a wheel of 35 teeth when (*a*) the circular pitch is $1\frac{1}{4}$ in., (*b*) the diametral pitch is $1\frac{3}{4}$, (*c*) the module pitch is 0·6.

14. Two parallel shafts, the axes of which are about 4 ft. apart, are to be connected by a pair of toothed wheels so that one shall rotate $3\frac{1}{2}$ times as fast as the other. If the diametral pitch is $1\frac{3}{4}$, find the number of teeth in each wheel and the exact distance between the centres of the shafts.

If the distance between the shaft centres is to be accurately 4 ft., what is the actual velocity ratio?

15. Two shafts whose axes are parallel are to be connected by a pair of toothed wheels. The velocity ratio between the shafts is to be 3·3 and the diametral pitch $2\frac{1}{2}$. The approximate distance between the centres of the shafts is 25 in. Find the number of teeth in each wheel and the actual distance apart of the centres of the two shafts.

If the distance between the centres is to be within $1\frac{1}{2}$ per cent. of 25 in., what modification of the diametral pitch and numbers of teeth is necessary so that the velocity ratio may still be 3·3?

16. What are the conditions which must be satisfied by the curves which form the profiles of wheel teeth in order that the angular velocity ratio between the wheels may be constant? Point out the advantages of involute teeth, and show how to set out the profile of the tooth for a given obliquity. [*Lond. B.Sc.*]

17. State and prove the essential condition that wheel teeth may give a constant velocity ratio. Two spur wheels have 40 and 60 teeth

of 1 in. pitch, the arcs of approach and recess each being $\frac{3}{4}$ in., and the flanks of the teeth radial. Determine the length of the addendum of the teeth on each wheel. [*Lond. B.Sc.*]

18. In a spiral gear drive the slope of the teeth on the driving-wheel A has been fixed at 50°. The normal pitch of the teeth is $\frac{1}{2}$ in. and the wheel A runs at twice the speed of the driven wheel B. The shafts are at right angles and the distance between their centres is approximately 7 in. Determine the dimensions of suitable wheels for this drive, giving for each wheel (a) the number of teeth, (b) the slope of the teeth, (c) the circular pitch, (d) the pitch diameter. Give also the correct distance between the wheel centres. [*Lond. B.Sc.*]

19. Find the number and pitch of the teeth of two toothed wheels to transmit a velocity ratio of 4 to 1 between two shafts whose centres are approximately $25\frac{5}{8}$ in. apart. The following conditions must be satisfied: (a) a standard diametral pitch must be chosen from amongst the values 1, $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, 2, $2\frac{1}{4}$, $2\frac{1}{2}$, 3, and $3\frac{1}{2}$; (b) the actual distance between the shaft centres must not vary more than 1 per cent. of that given; and (c) the number of teeth must be as small as possible. [*Lond. B.Sc.*]

20. A spur wheel of 120 teeth drives another spur wheel of 40 teeth, and the pitch of the teeth is 1 in. The flanks of the teeth of each wheel are radially straight and the arcs of approach and recess are each equal to the pitch. Find the heights of the teeth in the two wheels above the pitch circles. [*Lond. B.Sc.*]

21. The lay shaft of a two-speed gear box carries two gear wheels A and B. The propeller shaft, parallel to the lay shaft, carries two sliding gears C and D which can gear respectively with A and B to give two speeds to the propeller shaft. The distance between the shaft centres is $8\frac{1}{4}$ in. The speed of the lay shaft is 1,000 revs. per min. and the speed of the propeller shaft is to be exactly 650 revs. per min. when A and C are in gear and as nearly as possible 420 revs. per min. when B and D are in gear.

If the least number of teeth on any wheel is not to be less than 20 and the greatest number of teeth on the smaller wheels is not to exceed 40, find suitable tooth numbers and diametral pitches. [*Lond. B.Sc.*]

22. A shaft running at 600 revs. per min. carries two gear wheels A and B. A second shaft parallel to the first carries two sliding gears C and D which gear with A and B respectively to give two speeds to the second shaft. The speeds of the second shaft are to be 375 exactly and approximately 220 revs. per min. The distance between the shaft centres is approximately $6\frac{3}{4}$ in. and a diametral pitch of 4 can be used on all gears.

Find suitable teeth numbers for all gears and give the exact distance between the shafts. [*Lond. B.Sc.*]

CHAPTER VII

WHEEL COMBINATIONS AND EPICYCLIC GEARS

§ **100. Wheel Trains.** Two or more gear wheels gearing together constitute a wheel combination or train of wheels. The velocity ratio between two wheels in gear has already been investigated in the preceding chapter, and by an extension of this method the velocity ratio for a number of wheels in gear may be easily found. The velocity ratio, or ratio of the speed of the last wheel of the train to the first wheel, is often referred to as the value of the train or gear ratio.

Ordinary gear trains may be divided into two classes, simple and compound. Ordinary gear trains are those in which the *axes of the various gears are fixed relative to each other.* A simple train of gears is one in which each axis carries only one wheel; a compound train of gears has one or more axes on which two wheels may rotate together. In an epicyclic wheel train one or more of the axes may move relative to a fixed axis.

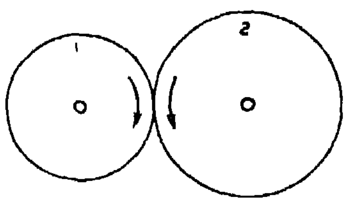


Fig. 70.

§ **101. Simple Trains of Wheels.** In Fig. 70 the two circles represent two wheels in gear, thus constituting a simple train. Assuming wheel 1 to be the driver, wheel 2 is the follower, and using the usual notation of ω for angular speed in radians per second, N for revolutions per minute, and T for number of teeth, the value of the train is $\frac{\omega_2}{\omega_1} = \frac{T_1}{T_2}$. In practice it is more convenient to work in revolutions per minute and

this ratio becomes $\frac{N_2}{N_1} = \frac{T_1}{T_2}$; the wheels rotate in opposite directions. In Fig. 71 we have a simple train of three wheels, from which

$$\frac{N_2}{N_1} = \frac{T_1}{T_2} \quad \text{and} \quad \frac{N_3}{N_2} = \frac{T_2}{T_3}.$$

$$\therefore \frac{N_3}{N_1} = \frac{T_1}{T_2} \times \frac{T_2}{T_3} = \frac{T_1}{T_3}.$$

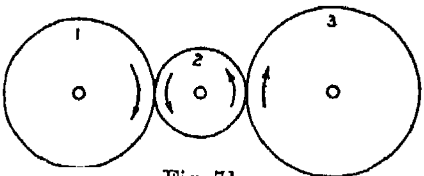


Fig. 71.

The velocity ratio is equal to the ratio of the number of teeth in the first and last wheels respectively, being independent of the number of teeth in the intermediate wheel. It will be noticed, however, that the last wheel rotates in the same direction as the first wheel. Wheel 2, having no influence upon the velocity ratio, is called an idle wheel. Any number of wheels may be placed between the first and last wheels without altering the velocity ratio; the direction of rotation of the last wheel will be the same as that of the driver if the number of intermediate wheels is odd, and of opposite direction if the number is even.

Let x stand for last wheel.

$$\frac{N_x}{N_1} = \frac{T_1}{T_2} \times \frac{T_2}{T_3} \times \frac{T_3}{T_4} \times \dots \times \frac{T_{x-2}}{T_{x-1}} \times \frac{T_{x-1}}{T_x} = \frac{T_1}{T_x}.$$

§ **102. Compound Train of Gears.** A compound train of gears is shown in Fig. 72, in which wheel 1 is the driver, wheels 2 and 3 form a compound wheel since they rotate together, and wheel 4 is the last wheel.

$$\frac{N_2}{N_1} = \frac{T_1}{T_2} \quad \text{and} \quad \frac{N_4}{N_3} = \frac{T_3}{T_4}, \quad \text{also } N_2 = N_3.$$

$$\therefore \frac{N_4}{N_1} = \frac{T_1}{T_2} \times \frac{T_3}{T_4}.$$

Wheel 1 being the driver, wheel 2 may be regarded as a follower, and wheels 3 and 4 may be regarded also as driver and follower respectively.

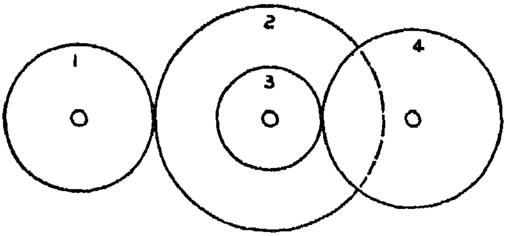


Fig. 72.

Then

$$\frac{N_4}{N_1}=\frac{T_1\times T_3}{T_2\times T_4} \quad \frac{\text{product of numbers of teeth on drivers}}{\text{product of numbers of teeth on followers}}.$$

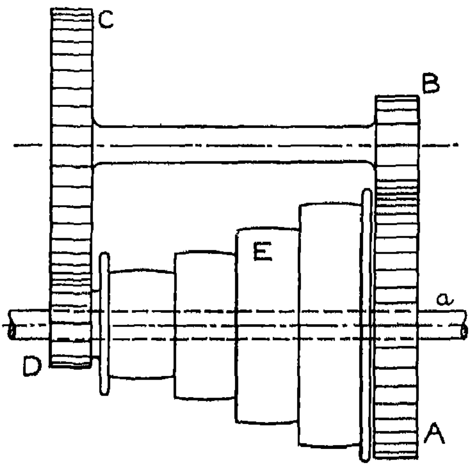


Fig. 73.

§ **103. Back-gear of Lathe Headstock.** The arrangement of the back-gear of a lathe headstock is an example of a compound train of wheels. In Fig. 73, a is the spindle of the lathe upon which is keyed a large gear wheel A. The cone pulley E, to which is attached a pinion D, can rotate freely on the spindle a. Gearing with A and D are two wheels B and

C mounted upon the same hollow shaft so that they rotate at the same speed; this shaft rotates on an arbor which has its ends turned eccentric with the axis, so that when the arbor is turned through a given angle the gears B and C may be disengaged from the gears A and D, in which case the back-gear is said to be 'out'.

The cone E is driven from a corresponding cone on the countershaft, and when the back-gear is out the cone pulley E is fastened to the wheel A by an adjustable bolt. With the four possible positions of the belt on the cone there are four spindle speeds. When the back-gear is 'in', the cone pulley E is disengaged from A and the drive is now from D to C and from B to A. There are four possible speeds with the back-gear in, making eight speeds in all. The velocity ratio when the back-gear is in is

$$\frac{N_A}{N_D}=\frac{\omega_A}{\omega_D}=\frac{T_D \times T_B}{T_C \times T_A}.$$

§ **104. All-geared Headstock.** Modern lathes are frequently designed with an all-geared headstock in which the use of cone pulleys is dispensed with. In Fig. 74 a diagrammatic plan arrangement is shown of an all-geared headstock, for giving twelve different spindle speeds. The main spindle is e on the axis c. An axis b is parallel to c, and in the actual headstock an axis a is above and parallel to the axis b; for convenience, in the figure, the axis a is shown in the same plane as b and c. Another axis d parallel to a is in the front of the headstock.

The gears A and B are fastened together and are driven at constant speed by a motor. The gears A and B can be moved along the axis of a so that A gears with C as shown, or B can gear with D, or they may be in neutral, i.e. neither A nor B is in gear. The gears C, D, E, F, and G are all keyed to the same shaft whose axis is b, and all the gears rotate together. The gears J, K, and L are keyed to a hollow sleeve f along which they can slide so that L may gear with F as shown, or K gear with G, or J gear with E. Keyed to the same sleeve is a gear wheel H which does not move along the axis. The

wheel I is keyed to the main spindle e and by means of a clutch I may rotate with H or it may rotate independently of H. The wheels M and N are equivalent to a 'back-gear' and may be thrown in or out of gear with H and I respectively; when M and N are in gear I is not clutched to H, but when M and N are out of gear I is clutched to H.

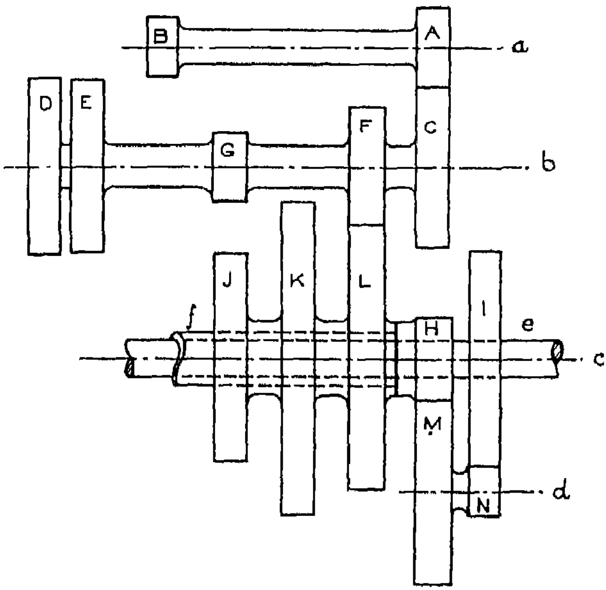


Fig. 74.

For the given position of the gears, the drive is from A to C; F to L; H to M; and N to I. If the gears M and N are out, the drive is from A to C; F to L and H, thence direct to I.

The gears A and B run at a constant speed of 400 revs. per min., hence for the given position of the gears, the speed of I, which is the speed of the main spindle e,

$$= 400 \times \frac{T_A}{T_C} \times \frac{T_F}{T_L} \times \frac{T_H}{T_M} \times \frac{T_N}{T_I}.$$

The numbers of teeth on the wheels are A, 26; B, 20; C, 54; D, 60; E, 59; F, 39; G, 23; H, 26; I, 51; J, 68; K, 104; L, 88; M, 62; and N, 15.

$\therefore$ speed of spindle $e = 400 \times \frac{20}{54} \times \frac{30}{88} \times \frac{20}{62} \times \frac{15}{51} = 10{\cdot}53$ revs. per min.

The spindle speed for other positions of the gears may be readily calculated in a similar way. These are shown in tabular form below.

Wheels in gear	*Speed of spindle e*	
ACEJ	$400 \times \frac{20}{54} \times \frac{50}{68}$	$= 167{\cdot}1$
BDEJ	$400 \times \frac{20}{60} \times \frac{50}{68}$	$= 115{\cdot}7$
ACFL	$400 \times \frac{20}{54} \times \frac{30}{88}$	$= 85{\cdot}35$
BDFL	$400 \times \frac{20}{60} \times \frac{30}{88}$	$= 59{\cdot}10$
ACGK	$400 \times \frac{20}{54} \times \frac{23}{104}$	$= 42{\cdot}60$
BDGK	$400 \times \frac{20}{60} \times \frac{23}{104}$	$= 29{\cdot}50$
ACEJHMNI	$400 \times \frac{20}{54} \times \frac{50}{68} \times \frac{20}{62} \times \frac{15}{51}$	$= 20{\cdot}60$
BDEJHMNI	$400 \times \frac{20}{60} \times \frac{50}{68} \times \frac{20}{62} \times \frac{15}{51}$	$= 14{\cdot}27$
ACFLHMNI	$400 \times \frac{20}{54} \times \frac{30}{88} \times \frac{20}{62} \times \frac{15}{51}$	$= 10{\cdot}53$
BDFLHMNI	$400 \times \frac{20}{60} \times \frac{30}{88} \times \frac{20}{62} \times \frac{15}{51}$	$= 7{\cdot}29$
ACGKHMNI	$400 \times \frac{20}{54} \times \frac{23}{104} \times \frac{20}{62} \times \frac{15}{51}$	$= 5{\cdot}25$
BDGKHMNI	$400 \times \frac{20}{60} \times \frac{23}{104} \times \frac{20}{62} \times \frac{15}{51}$	$= 3{\cdot}64$

With the 'back-gear' out there are three speeds when A gears with C and three when B gears with D. Similarly, there are six speeds with 'back-gear' in, giving twelve speeds in all.

§ 105. Gear Trains for Screw-cutting Lathes. A screw-cutting lathe may be used for cutting screw threads on a shaft or spindle. The given shaft is placed between the lathe centres and made to rotate at the same speed as the main spindle of the lathe. The saddle of the lathe, to which is rigidly attached the cutting tool, is caused to move at a given speed by means of a split nut on the saddle engaging with a screw which rotates; this screw is termed the lead screw, and in length is almost equal to the length of the lathe bed. The lead screw is geared to the main spindle by gearing, and in order to cut a variety of threads this gearing must be capable of being altered in such a manner as to give the required velocity ratio between the main spindle and the lead screw. Screw threads may be right or left hand, and accordingly the lathe must be capable of producing either of these.

The lead screw may be right or left hand, usually right, and to cut a right-hand screw the cutting tool must travel towards the fast headstock as the work in the lathe rotates. To cut a left-hand screw the cutting tool must travel in the opposite direction. A right-hand lead screw requires to turn in the same direction as the machine spindle to cut a right-hand thread. To construct sets of gears which will give any required velocity ratio between the main spindle and the lead screw, a set of change wheels is usually provided. A set of change wheels usually has wheels with 20, 25, 30, etc., teeth, going up in 5's to about 120. Very often two 60- or two 50-toothed wheels are supplied, as these facilitate the construction of a gear train.

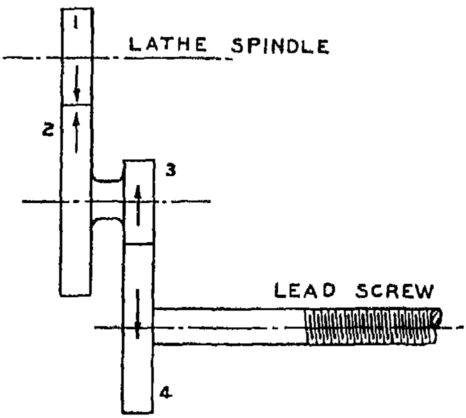


Fig. 75.

In Fig. 75, let wheels 1, 2, 3, and 4 represent a compound train of wheels, arranged for cutting screw threads. Wheel 4 is attached to the lead screw, wheel 1 to the main spindle, wheels 2 and 3 are compound.

Let n = number of threads per inch to be cut,

l = number of threads per inch on lead screw.

Whilst the cutting tool travels one inch the lead screw must make l revolutions and the main spindle n revolutions. Hence l revolutions of the lead screw are made in the same time as n revolutions of the main spindle. Hence the velocity

ratio between the lead screw and the main spindle must be $\frac{l}{n}$; but this velocity ratio is $\frac{N_4}{N_1}$ or $\frac{T_1 \times T_3}{T_2 \times T_4}$; hence $\frac{l}{n} = \frac{T_1 \times T_3}{T_2 \times T_4}$ = velocity ratio of lead screw to main spindle.

Any convenient combination of gears in which $\frac{T_1 \times T_3}{T_2 \times T_4} = \frac{l}{n}$ will cut n threads per inch if the lead screw has l threads per inch. The thread will be left or right hand according to whether the lead screw is left or right hand. Referring to Fig. 75, wheel 4 rotates in the same direction as the lathe spindle, and for a right-hand lead screw the lathe will cut a right-hand thread. To cut a left-hand thread an idle wheel may be interposed between wheels 1 and 2 or between wheels 3 and 4.

EXAMPLE 1. Find a suitable train of wheels to cut 24 threads per inch right hand, if the lead screw has 4 threads per inch right hand.

Required velocity ratio $= \frac{4}{24} = \frac{1}{6}$,

$\therefore \frac{T_1 \times T_3}{T_2 \times T_4} = \frac{1}{6}$. T_1 may be made equal to 20. T_2 may be made equal to 60.

Hence $\frac{T_3}{T_4} = \frac{1}{2}$, and any two wheels whose numbers are in this ratio may be used, e.g. 40 and 80.

EXAMPLE 2. Find a suitable train to cut 14 threads per inch left hand on a lathe which has a right-hand lead screw of 2 threads per inch.

Required velocity ratio $= \frac{2}{14} = \frac{1}{7}$,

$\therefore \frac{T_1 \times T_3}{T_2 \times T_4} = \frac{1}{7}$, from which $T_1 = 20$, $T_2 = 70$, $T_3 = 30$, and $T_4 = 60$ will satisfy the condition. Since the thread is left hand an idle wheel must be used between the wheels 1 and 2 or between 3 and 4.

If the number of threads to be cut is small, say two or three per inch, a single train of wheels may be sufficient to give the required velocity ratio, care being taken to use an idler or two idlers as may be necessary to ensure the required direction of rotation of the lead screw.

§ **106. Epicyclic Gears.** The problem of finding the velocity ratio when a number of wheels are geared together has already been dealt with, and providing the axes of the wheels are fixed relative to one another, this problem does not present any peculiar difficulty. When, however, one of the axes moves relative to a fixed axis, the resulting motion is much more difficult to visualize and the comparatively simple methods

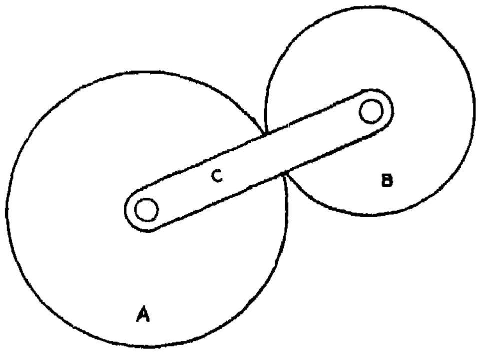


Fig. 76.

applied to ordinary simple or compound gears do not apply. In an epicyclic train of gears one wheel is usually fixed, but not necessarily so; it may be given a speed in either direction about its axis.

In Fig. 76, A and B represent two wheels in gear. For this to constitute an epicyclic train the axis of one wheel must move relative to the other. Assume the axis of A fixed and that the axis of B can rotate about the axis of A. This is conventionally shown by connecting the two axes with an arm C which is capable of rotation about the axis of A.

The problems connected with epicyclic gearing are, perhaps, better understood if the student has a clear conception of the meaning of relative motion. The relative motion between any two bodies is not affected by any motion they may have in common or by any motion added to or abstracted from both simultaneously. In Fig. 76 the relative motion between B and the arm is not affected by any motion given to the arm, since the wheel B must move as the arm moves. Consider first

the case when the arm is fixed and A makes one revolution, say clockwise, then B makes $\frac{T_A}{T_B}$ revolutions in the opposite direction, that is, in a counter-clockwise direction. The direction of rotation is of great importance in epicyclic gear problems and it is usual to give a positive sign to one direction and negative sign to indicate motion in the opposite direction. Thus A makes $+1$ revolution and B makes $-\frac{T_A}{T_B}$ revolutions. It is obvious that if A makes $+10$ revolutions then B makes $-10\frac{T_A}{T_B}$ revolutions. Returning to the original state in which A has made $+1$ revolution the relative motions between A, B, and C will not be altered if they are each given -1 revolution. This latter motion is common to all and does not in any way alter the relative motions between A, B, and C. The physical interpretation of giving each -1 revolution is that the whole gear is locked and turned as a whole -1 revolution. The net effect, now, is that A has been given $+1-1$ revolutions, and has virtually made no revolutions as far as the final condition is concerned, B has made $-\frac{T_A}{T_B}-1$ revolutions and C, -1 revolution.

Again considering the arm as fixed and when A has made $+10$ revolutions, B having made $-10\frac{T_A}{T_B}$ revolutions, adding -10 revolutions to each will not affect the relative motions, and performing this operation, it is seen that A has virtually made no revolutions, B has made $\left(-10\frac{T_A}{T_B}-10\right)$ revolutions, and C, -10 revolutions. This same result is obtained if the final conditions for A, B, and C, when A has been given $+1-1$ revolutions, is multiplied by 10. These results are more conveniently interpreted when expressed in tabular form.

In the following table, line 1 shows the condition in which C is fixed and A is given $+1$ revolution. Line 2 is obtained by multiplying line 1 by 10. Line 3 is obtained by adding -1 to line 1, and line 4 by multiplying line 3 by 10. Line 5 is

Line	*Condition*	*Revs. of A*	*Revs. of B*	*Revs. of C*
1	C fixed, $A+1$ rev.	$+1$	$-\frac{T_A}{T_B}$	0
2	C fixed, $A+10$ revs.	$+10$	$-10\frac{T_A}{T_B}$	0
3	A fixed, $C-1$ rev.	$+1-1=0$	$-\frac{T_A}{T_B}-1$	-1
4	A fixed, $C-10$ revs.	0	$-10\frac{T_A}{T_B}-10$	-10
5	A fixed, $C-10$ revs.	$+10-10=0$	$-10\frac{T_A}{T_B}-10$	-10

obtained by adding -10 to line 2. Lines 4 and 5, which agree, have been arrived at by different processes. It appears from the above analysis that having once determined the relative motions between the wheels and the arm, any number may be added to the results already obtained and they may be multiplied by any number.

§ **107. The Epicyclic Wheel.** In Fig. 76 the wheel B is known as the epicyclic wheel, since all points on the circumference describe an epicyclic path. When a wheel such as B rolls inside an annular wheel, points on the circumference of B describe hypocycloid paths, but the term epicyclic gear is employed to include all gears in which one of the axes rotates about a fixed axis.

There is one important distinction between epicyclic trains and ordinary gear trains, and that is, the number of revolutions of a wheel B, Fig. 76, is the number of revolutions in space and not about its axis. From the above table it is seen that when A is fixed and the arm C makes -1 revolution, B makes $-\frac{T_A}{T_B}-1$ revolutions; this is the number of revolutions made in space or relative to the fixed axis of A; the number of revolutions of B about its own axis is $-\frac{T_A}{T_B}$. This point is perhaps made clearer on reference to Fig. 77, in which C is an arm capable of rotating about the fixed axis O, and B is a wheel which can rotate about an axis at the other extremity

of C. Let the wheel B be fixed to the arm C so that no relative movement may take place. To identify the position of the wheel relative to the arm an arrow is marked on B in line with the arm C. When the arm turns from the position 1 to position 2, it has made $\frac{1}{4}$ revolution and the wheel B has also made $\frac{1}{4}$ revolution, but not about its own axis, since B is considered fixed to the arm. Moving the arm through the successive positions 3 and 4 to 1 again, it is seen that while the arm C has made 1 revolution the wheel B has also made 1 revolution in the same direction as the arm. This simple

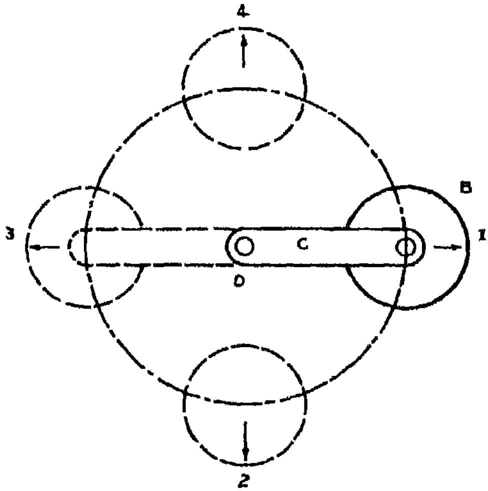


Fig. 77.

analogy accounts for the revolutions of B in Fig. 76 being increased (numerically) by 1 compared with the revolutions made when the arm is stationary.

Example 3. An arm which can rotate carries three wheels whose centres lie on a straight line. The first wheel has 40 teeth, the second 50, and the third 60. If the arm rotates once about the centre of the first wheel which is fixed, find the number of revolutions of the last wheel. If, instead of being fixed, the first wheel makes 10 revolutions in a direction opposite to the arm, while the latter makes 1 revolution, find the number of revolutions of the last wheel.

The train of gears is illustrated in Fig. 78, in which A, B, and

C correspond to the first, second, and third wheels respectively, and D is the arm. Constructing a table as already explained, columns A, B, C, and D represent the number of revolutions of wheels A, B, C, and the arm D respectively.

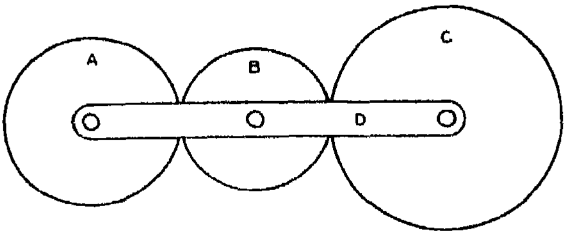


Fig. 78.

Line 1 is obtained by considering the simplest condition of the gears in which the arm is fixed.

Line 2 is obtained by adding -1 to line 1, giving the number of revolutions of $C = -\frac{1}{3}$. The negative sign indicates that C rotates in the same direction as D since the latter makes -1 revolution.

Line 3 is merely a statement of the required conditions for the second part of the problem, and remembering that we can multiply any line by any number and can add any number to any line, line 4 is obtained by multiplying line 1 by 11. Line 5 is obtained by adding -1 to line 4, giving the revolutions of $C = +\frac{19}{3}$, i.e. $\frac{19}{3}$ revolutions in a direction opposite to the arm.

Line	*Condition*	A	B	C	D
1	D fixed, $A+1$ rev.	$+1$	$-\frac{40}{50}$	$-\frac{40}{50}\times-\frac{50}{60}=+\frac{2}{3}$	0
2	A fixed, $D-1$ rev.	$+1-1$	$-\frac{4}{5}-1$	$+\frac{2}{3}-1=-\frac{1}{3}$	-1
3	$A+10$ revs., $D-1$ rev.	$+10$	?	?	-1
4	$A+11$ revs., D fixed	$+11$	$-\frac{4}{5}\times 11$	$+\frac{2}{3}\times 11$	0
5	$A+10$ revs., $D-1$ rev.	$+11-1=+10$	$-\frac{44}{5}-1=-\frac{49}{5}$	$+\frac{22}{3}-1=+\frac{19}{3}$	-1

§ **108. Alternative Method of Solution.** In the construction of a table a little difficulty is experienced sometimes in deciding the multiplying factor required to give one of the wheels the correct number of revolutions. This difficulty is, perhaps, more pronounced when one of the wheels, instead of being fixed, rotates at a given speed.

In § 106 it has already been seen that having once determined the various motions of the wheels when the arm is fixed, any number may be added to the results already obtained and they may be multiplied by any number. A general method of solution is to fix the arm and find the revolutions made by the different wheels when any one wheel is given one revolution. Multiplying this line by x simply implies that instead of giving any one wheel one revolution it is given x revolutions; it follows that all other wheels make x times as many revolutions as for one revolution of the one wheel. The next step is to add y revolutions to the result already obtained by multiplying by x. The interpretation of this step is that the whole gear is locked and given y revolutions. The revolutions made by the different wheels are thus expressed in terms of x and y. If one wheel is fixed, then the expression representing its revolutions is equated to zero and gives one relation between x and y. A second relation is found from the known speed of a driving or driven wheel. From these two equations values of x and y can be found and thus the revolutions made by other wheels in the epicyclic gear can be determined. To illustrate this method Example 3 will be worked out.

A	B	C	D
$+1$	$-\frac{40}{50}$	$+\frac{2}{3}$	0
$+x$	$-\frac{40}{50}x$	$+\frac{2}{3}x$	0
$+x+y$	$-\frac{4}{5}x+y$	$+\frac{2}{3}x+y$	$+y$

In constructing the table the respective speeds of A, B, and C are found when the arm D is fixed and A given $+1$ revolution. The second line is found by multiplying the first line by x; the third line by adding $+y$ to the second line.

In the first part of the problem A is the fixed wheel and hence $x+y=0$; the arm makes 1 revolution, hence $y=+1$.

Evaluating for x, $x=-1$.

Hence speed of $C=\frac{2}{3}x+y=-\frac{2}{3}+1=+\frac{1}{3}$.

Thus the wheel C makes $+\frac{1}{3}$ revolution while the arm makes $+1$ revolution, i.e. they rotate in the same direction.

In the second part of the problem A makes 10 revolutions

in a direction opposite to the arm and the following equations are thus formed

$$y = +1$$

$$x+y = -10.$$

Hence

$$x = -11.$$

$$\therefore \text{ Revolutions of } C = \tfrac{2}{3}x+y = \tfrac{2}{3}\times -11+1$$

$$= -\tfrac{19}{3}.$$

Thus C makes $\frac{19}{3}$ revolutions in a direction opposite to that of the arm for one revolution of the latter.

§ **109. Reverted Train.** When the axis of the last wheel coincides with the axis of the first wheel of an epicyclic train, the combination is called a reverted train. Such a train is shown in Fig. 79, in which the arm E can rotate about either the axis of A and D or about the axis of B and C. If the arm rotates about the axis of A and D, then either A or D may be fixed, or they may both be moving. The wheels B and C are compound and rotate together. Reverted trains can be used when exceptionally large or small velocity ratios are required.

§ **110. Odometer or Cyclometer.** The gearing of an odometer, which is an instrument for measuring and recording the distance travelled by ordinary pedal cycles, is illustrated in Fig. 80, and is an example of a reverted train. This is a very compact mechanism, the whole gear being enclosed in a cylinder about $\frac{3}{4}$ in. diameter. A projecting striker, fixed on a spoke of the cycle wheel, rotates with the cycle wheel, and comes in contact with a projection or tooth on the star wheel once during each revolution of the cycle wheel. The star wheel has eight teeth and thus rotates $\frac{1}{8}$ revolution per revolution of the cycle wheel. This part of the mechanism is not shown in Fig. 80. The star wheel is integral with the arm E which can rotate about the axis of the wheels C and A. The wheels B and D are compound and rotate together about the other extremity of E. The wheels B and D gear internally with the annular wheels A and C respectively, of which A is fixed and C capable of rotation about its axis. On the outside of the wheel C, the numbers 1, 2, 3, . . . 9, 0, are engraved and a window is fixed in the cylinder in such a manner that only

one figure is visible at any particular position of C. A movement of one figure to the next indicates that $\frac{1}{10}$ of a mile has

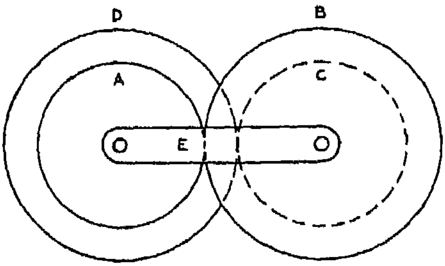


Fig. 79.

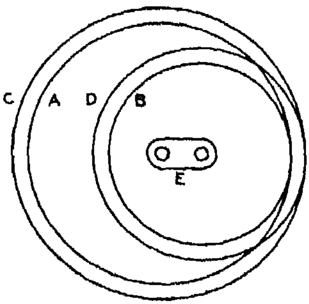


Fig. 80.

been traversed, so that C makes one complete revolution per mile traversed by the cycle wheel. Taking a 28-inch diameter cycle wheel, the wheel is required to make $\frac{1,760 \times 36}{\pi \times 28}$ or 720 revolutions per mile. The star wheel makes $\frac{1}{8}$ revolution per revolution of the cycle wheel, and since the arm E is integral with the star wheel, E is required to make $\frac{720}{8}$ or 90 revolutions per mile.

A	B and D	C	E
$+1$	$+\frac{24}{21}$	$+\frac{24}{21}\times\frac{23}{26} = +\frac{92}{91}$	0
$+x$	—	$+\frac{92}{91}x$	0
$+x+y$	—	$+\frac{92}{91}x+y$	$+y$

In a particular gear, A is fixed and has 24 internal teeth, B has 21, C internal 26, and D 23 teeth. To find the actual number of revolutions made by E for one revolution of C, a table is constructed in the usual manner. This is shown in the above table, and since the number of revolutions of B and of D is not required, it is usual to leave this column blank after the first line.

When the cycle travels 1 mile wheel C makes one revolution.

Hence $$\frac{92}{91}x+y = 1.$$

Also since A is the fixed wheel, $x+y = 0$.

Hence $$x = -y$$

and $$\frac{92}{91}y + y = 1$$

or $$y = -91.$$

In the construction of the table it must be remembered that wheels gearing internally rotate in the same direction. From these results it is seen that E makes 91 revolutions per revolution of C, instead of 90 as calculated from the diameter of the cycle wheel. This error is slightly more than 1 per cent., but when it is remembered that it is extremely improbable that the radius of the tyre at the point of contact with the ground is exactly 14 inches, this error is not excessive; also the radius from the centre of the cycle wheel to the ground depends upon the degree of inflation of the tyre and upon the weight of the rider.

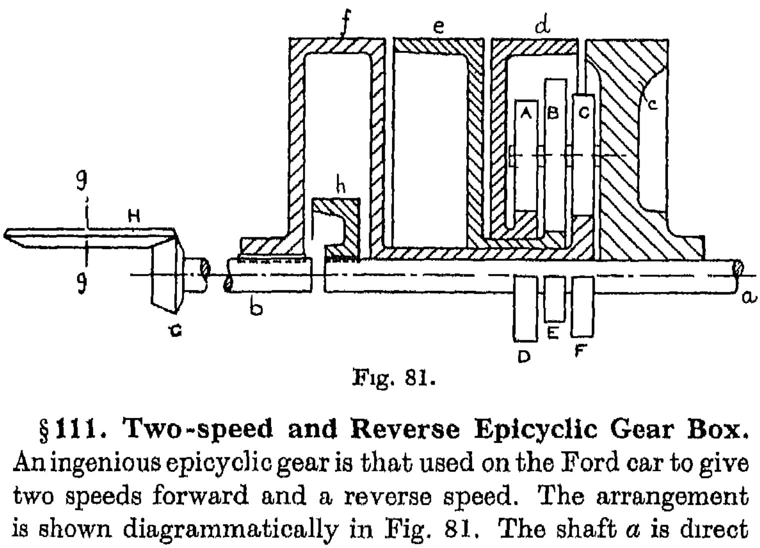


Fig. 81.

§ **111. Two-speed and Reverse Epicyclic Gear Box.** An ingenious epicyclic gear is that used on the Ford car to give two speeds forward and a reverse speed. The arrangement is shown diagrammatically in Fig. 81. The shaft a is direct from the engine and rotates at engine speed. Keyed to this shaft is the fly-wheel c. Three pins spaced at equal intervals on the fly-wheel each carry three gear wheels A, B, and C which are compounded so that they all rotate together. In the figure only one set of gear wheels A, B, and C is shown to avoid confusion. The epicyclic wheels A, B, and C gear

respectively with three wheels D, E, and F whose axes coincide with the axis of the engine shaft a. The gear wheel F is connected directly through the drum f to the shaft b upon which is keyed a bevel pinion G, gearing with another bevel H which is connected through differential gearing (not shown) to the rear axle g. The wheels D and E are connected to drums d and e respectively, around which brake bands are fitted, so that either of the drums, and consequently either of the wheels D and E, may be clamped or locked.

The numbers of teeth on the wheels are A 24; B 33; C 27; D 30; E 21; F 27; G 22; and H 80. For the present, the effect of the differential gear will be neglected as this will be explained later, so that the road wheels, which are attached to the back axle, may be assumed to rotate at the same speed as the bevel wheel H.

Line	*Condition*	*A, B, and C*	*F*	*E*	*D*	*Arm*
1	Arm fixed, $F+1$ rev.	$-\frac{27}{27} = -1$	$+1$	$-1 \times -\frac{33}{21} = +\frac{33}{21}$	$-1 \times -\frac{24}{30} = +\frac{24}{30}$	0
2	E fixed, arm $-\frac{33}{21}$ revs.	..	$+1-\frac{33}{21} = -\frac{12}{21}$	0 ..	..	$-\frac{33}{21}$
3	D fixed, arm $-\frac{24}{30}$ revs.	..	$+1-\frac{24}{30} = +\frac{6}{30}$	..	0	$-\frac{24}{30}$

In the above table the speeds of F and the arm (or flywheel) have been found when the wheels E and D have been clamped in turn.

For top gear the drive is direct, as neither D nor E is locked, and the shaft b is connected through clutch plates h and f to the engine shaft a. The wheel F is also connected to shaft b through the drum f so that F rotates at engine speed and the gear does not operate as an epicyclic gear, the gears C and F rotating as one; in other words, A, B, and C do not rotate upon their own axes. The ratio of road-wheel speed to engine speed is thus the ratio of the numbers of teeth in the bevels G and H.

$$\frac{\text{road-wheel speed}}{\text{engine speed}} \quad +\frac{22}{80} = +\frac{11}{40}.$$

For second gear, the clutch is disengaged and the drum e

which is connected to the wheel E is clamped by an encircling band: in this case it is seen from the table that F makes $-\frac{12}{21}$ revolution while the arm (or engine shaft) makes $-\frac{33}{21}$ revolution. The drive is through F to shaft b.

$$\frac{\text{road-wheel speed}}{\text{engine speed}} = +\frac{12}{33} \times \frac{22}{80} = +\frac{1}{10}.$$

For reverse, the clutch is disengaged and the drum d which is connected to D is clamped by an encircling band; when this occurs F makes $+\frac{6}{30}$ revolution while the arm makes $-\frac{24}{30}$:

$$\frac{\text{road-wheel speed}}{\text{engine speed}} = -\frac{6}{24} \times \frac{22}{80} = -\frac{11}{160}.$$

The negative sign indicates that the direction of rotation of the road wheels is opposite to that when on first and second gears.

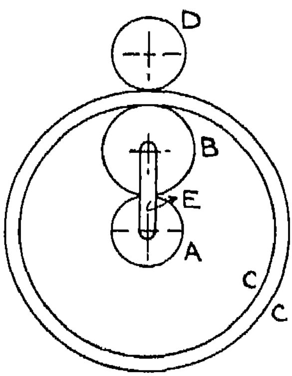


Fig. 82.

§ **112. Variable Speed-reduction Gear.** An interesting variable-speed gear used in the Italian Navy for gun elevating and training gear is shown diagrammatically in Fig. 82. The pinion A is keyed to a motor shaft which runs at constant speed. The wheel B gears externally with A and is free to

rotate on a pin carried by the arm E keyed to the driven shaft. The wheel B also gears internally with a wheel C concentric with the motor shaft but capable of rotating independently of the latter. The wheel C also has external teeth and is driven by a pinion D which can be driven at variable speed from the motor through friction gearing which is not shown. The friction gearing is designed so that D rotates in the same direction as the motor shaft. The axis of the wheel D is fixed relative to the axis of the motor shaft and consequently D is a gear wheel *external* to the epicyclic gear and care must be exercised in the construction of a table of respective speeds that D is not included in this table, since its speed is determined by the position of the friction disks. In Example 4 which follows, a table is constructed showing the speeds of A, C, and E. The first line is obtained by considering the arm at rest and A is given $+1$ revolution. The second line is obtained by multiplying the first line by x, the third line by adding y to the second line.

EXAMPLE 4. In the gear shown in Fig. 82 number of teeth on A = number of teeth on $D = \frac{1}{4} \times$ number of teeth on C.

Wheel C has the same number of teeth internally and externally.

Find (a) the speed of the driven shaft when D makes 800 revolutions per minute in the same direction as the motor shaft whose speed is 500 revolutions per minute;

(b) the speed and direction of rotation of D when the driven shaft rotates at the same speed as in (a) but in the opposite direction.

Omitting the revolutions made by wheel B, and remembering that D is external to the epicyclic gear, the table of speeds is as shown.

A	C	E
$+1$	$-\frac{1}{4}$	0
$+x$	$-\frac{1}{4}x$	0
$+x+y$	$-\frac{1}{4}x+y$	$+y$

(a) Speed of $A = +500$.

Hence $x+y = +500.$

$$\text{Speed of } D = +800.$$

$$\therefore \text{ Speed of } C = -\tfrac{1}{4}\times 800 = -200.$$

Hence $$-\tfrac{1}{4}x+y = -200,$$

from which $$x = 560$$

$$y = -60.$$

Hence the driven shaft rotates at 60 revolutions per minute in a direction opposite to that of the motor shaft.

(*b*) Speed of driven shaft is 60 revolutions per minute in the same direction as motor shaft; hence

$$y = +60$$

$$x+y = 500.$$

$$\therefore x = 440.$$

$$\text{Speed of } C = -\tfrac{1}{4}x+y = -\tfrac{1}{4}\times 440+60$$

$$= -50.$$

$\therefore$ Speed of $D = -50\times -4 = +200.$

Hence the speed of D must be 200 revolutions per minute in the same direction as the motor shaft.

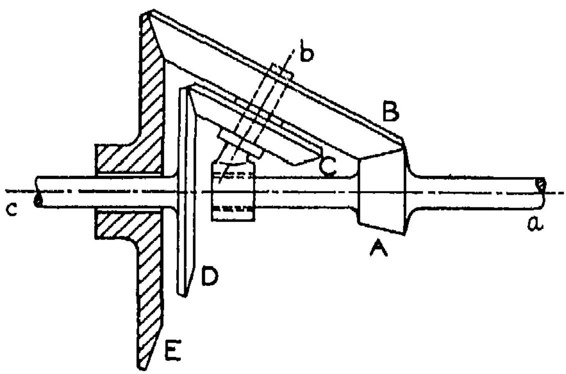


Fig. 83.

§113. Bevel Epicyclic Gears—Humpage's Gear. The methods applicable to the solution of epicyclic trains of spur gears are also valid for bevel epicyclics. A common type of bevel epicyclic gear is that known as Humpage's gear, which has been used with success on lathe headstocks, capstan lathes, and some types of electrical machines. In Fig. 83, *a*

is the driving shaft on which is keyed a bevel pinion A. E is a fixed wheel and D is fastened to the driven shaft c. Bevel wheels C and B are compound and gear respectively with the bevels D and E. The wheels C and B are free to rotate on an arm b inclined to the axis of a and c; the arm is also free to rotate about the axis of a and c. In actual gears two such wheels as B and C are used, the other pair being on the opposite side of the axis of a and c.

EXAMPLE 5. In the Humpage's gear shown in Fig. 83 the numbers of teeth on the wheels A, B, C, D, and E are 20, 64, 30, 50, and 80 respectively. If the shaft a rotates at 400 revs. per min., find the speed of shaft c.

In the formation of the table showing the relative speeds it may be noted that the wheels B and C rotate in a different plane from the wheels A, D, and E; hence it is not usual to affix any sign to the direction of rotation of these wheels, although they influence the direction of rotation of the other wheels.

In the above table the arm is first of all considered fixed and A given $+1$ revolution; this gives line 1, from which E makes $-\frac{1}{4}$ revolution. Line 2 is obtained by adding $+\frac{1}{4}$ revolution throughout, and in this line it is seen that A makes $+1\frac{1}{4}$ revolutions and

Line	*Condition*	*A*	*B and C*	*D*	*E*	*Arm*
1	Arm fixed, A $+1$ rev.	$+1$	$\frac{20}{64}$	$-\frac{20}{64}\times\frac{30}{50}$	$-\frac{20}{64}\times\frac{64}{80}$ $=-\frac{1}{4}$	0
2	E fixed, arm $+\frac{1}{4}$ rev.	$+1+\frac{1}{4}$	..	$-\frac{3}{16}+\frac{1}{4}=+\frac{1}{16}$	0	$+\frac{1}{4}$
3	E fixed, A 400 revs.	$+\frac{5}{4}\times\frac{4}{5}\times 400$	..	$+\frac{1}{16}\times\frac{4}{5}\times 400$	0	..

D $+\frac{1}{16}$ revolution. Line 3 is obtained by multiplying line 2 by the number required to convert $1\frac{1}{4}$ into 400. This number is $\frac{4}{5}\times 400$. Hence the number of revolutions of D is $+\frac{1}{16}\times\frac{4}{5}\times 400$ or 20.

EXAMPLE 6. In the Humpage's gear shown in Fig. 83, the wheel E, instead of being at rest, makes -40 revs. per min., while the shaft a is driven at $+400$ revs. per min. The numbers of teeth on the wheels are the same as in Example 5. Find the speed of D.

In the construction of the table, the arm is first considered as fixed and the wheel A is given $+1$ revolution. This gives line 1. Line 2 is obtained by multiplying line 1 by x, and line 3, by

Line	Condition	A	B and C	D	E	Arm
1	Arm fixed, $A+1$ rev.	$+1$	$\frac{20}{64}$	$-\frac{20}{64}\times\frac{30}{50} = -\frac{3}{16}$	$-\frac{20}{64}\times\frac{64}{80} = -\frac{1}{4}$	0
2	Arm fixed, $A+x$ revs.	$+x$	..	$-\frac{3}{16}x$	$-\frac{1}{4}x$	0
3	All given $+y$ revs.	$+x+y$	..	$-\frac{3}{16}x+y$	$-\frac{1}{4}x+y$	$+y$

adding $+y$ to line 2. From line 3, the number of revolutions of A is $+x+y$, which is equal to $+400$; and the number of revolutions of E is $-\frac{1}{4}x+y$, which is equal to -40.

$$\therefore\ x+y = 400$$
$$-\tfrac{1}{4}x+y = -40$$
$$\tfrac{5}{4}x = 440 \qquad \text{or } x = 352.$$
$$\therefore\ y = 48.$$

The number of revolutions of D is $-\frac{3}{16}x+y$, hence substituting values of x and y,

$$\text{speed of } D = -\tfrac{3}{16}\times 352+48 = -18.$$

The negative sign indicates that D rotates in the same direction as E and in the opposite direction to A

§ **114. Differential Gear.** In motor vehicles the rear wheels are usually the driving wheels. The front wheels can rotate freely, each on its own axis, and when the car is rounding a curve each of the front wheels can readily adapt itself to the conditions. The outer wheels must travel farther than the inner wheels when rounding a curve, and since both rear wheels are driven by the engine through gearing, some automatic device is required whereby the two rear wheels are driven at slightly different speeds. This is accomplished by the differential gear which is fitted on the rear axle. Differential gears are used on other machines, such as Port and Starboard indicators, but the application to the rear axles of motor-cars is, perhaps, the best known.

Fig. 84 shows a differential gear as applied to the rear axle of a motor-car. The shaft a is driven by the engine through the gear box. Keyed to a is a bevel pinion A which gears with the bevel wheel B. The shafts b and c form the rear axle and the two rear driving-wheels are fixed to these shafts. Keyed to the shafts b and c are two equal bevels C and D which gear with the equal bevel pinions E and F, which are

free to rotate on their respective axes. The wheel B carries two brackets and the bevel pinions E and F are supported from these brackets.

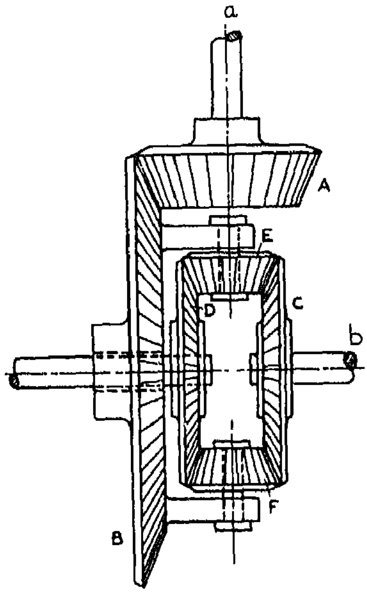


Fig. 84.

When the car is running in a straight path the bevel pinion A drives the bevel wheel B which carries the whole differential round as one unit and the wheels C and D rotate in the same direction as B and at the same speed. The bevel pinions E and F remain stationary on their own axes in this case. When rounding a curve the inner wheel has to travel a less distance than the outer wheel unless slip occurs, and it will be seen on reference to the following table that when one road wheel is decreased in speed, the other road wheel is increased in speed by a similar amount.

Line	*Condition*	*B*	*E and F*	*C*	*D*
1	B fixed, $D+1$ rev.	0	$\frac{T_D}{T_E}$	-1	$+1$
2	B fixed, $D+x$ revs.	0	..	$-x$	$+x$
3	All given $+y$ revs.	$+y$	..	$-x+y$	$+x+y$

The table is constructed in the usual manner and it is seen on reference to the table that the speed of B is the arithmetical mean of the speeds of C and D. Thus, if the road wheel on the shaft c experiences a resistance which decreases its speed, then the road wheel on shaft b is correspondingly increased in speed.

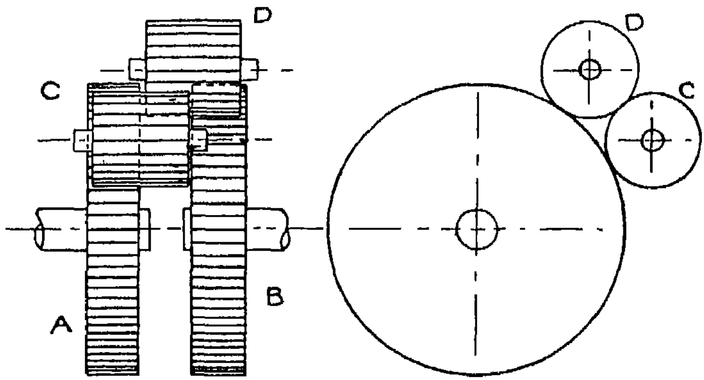


Fig. 85.

§ **115. Spur Differential Gear.** Another form of differential gear for the rear axle of a motor-car is the spur differential which has replaced the bevel differential. The gear is shown in Fig. 85. The gear wheels A and B are keyed to the two rear axles on which are keyed the road wheels. A pinion C gears with A and with a second pinion D. The pinion D gears with B. The pinions C and D can rotate freely on their own axes, but these axes are carried by a worm wheel and this constitutes the arm. The worm wheel is driven by a worm keyed to the shaft from the gear box. In practice four such pairs of pinions as C and D are used. This gear may be analysed in the same way as the bevel differential, and it will be found that *the arithmetical mean of the speeds of A and B is always equal to the speed of the arm carrying the pinions C and D, this latter speed being that of the worm wheel.*

§ **116. Speed of Epicyclic Wheel.** As explained in § 107 the revolutions of the epicyclic wheel (i.e. the wheel rotating on a pin carried by an arm) are not those about its

own axis but about the axis of the arm. Occasionally it is required to find the number of revolutions made by the epicyclic wheel about its own axis. Referring to Fig. 77, let the arm C be fixed and the wheel B be given $+x$ revolutions Next let B be locked to C and the whole be given $+y$ revolutions, i.e. in the same direction as the x revolutions. Then the total number of revolutions made by B about the axis of $C = x+y$.

$$\text{Hence } x = \text{total revolutions of } B-y,$$

i.e. revolutions of B about its own axis = total revolutions about axis of arm—revolutions of the arm.

§ **117. Sun and Planet Gear.** This gear, shown in Fig. 86, consists of an annular wheel A with internal teeth

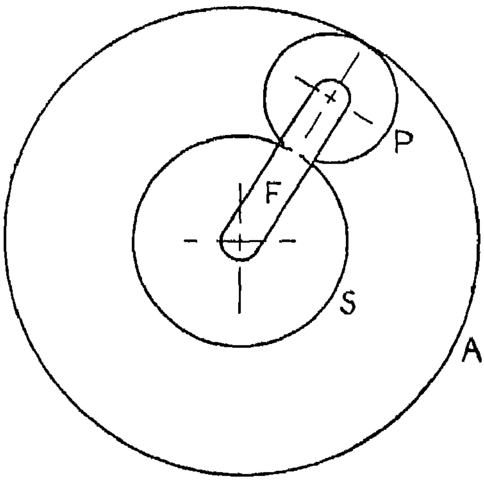


Fig. 86.

gearing with a planet wheel P which can rotate freely on an arm F. The sun wheel S gears with P and in the general case F, A, and S are free to rotate independently of each other, but in particular cases either A or S may be fixed and prevented from rotating. The driving member may be S or A, depending on whether A or S is fixed, and the arm F is the driven member. When three or more gears of this kind are built up in series a change-speed gear-box of the pre-

selective type can be developed, and the sun and planet gear may be regarded as a unit in such a gear-box.

In Fig. 86 let A and S represent the tooth numbers on A and S respectively.

Let M = speed of A in revolutions per minute.

N = ,, S ,, ,, ,,

To form a table showing the respective speeds, let the arm F be initially fixed and give $A+1$ revolution. Then the revolutions of S are $-\frac{A}{S}$. If A be given x revolutions, then S makes $-\frac{A}{S}x$ revolutions. Locking the whole gear and giving it y revolutions, the resulting speeds of A, F, and S are found in terms of x and y and shown in the following table:

A	S	F
$+1$	$-\frac{A}{S}$	0
$+x$	$-\frac{A}{S}x$	0
$x+y$	$-\frac{A}{S}x+y$	$+y$

Since the speed of A is M revs. per min. and the speed of S is N revs. per min., then

$$x+y = M$$

and

$$-\frac{A}{S}x+y = N.$$

$$x\left(1+\frac{A}{S}\right) = M-N.$$

$$x = \frac{M-N}{A+S}S$$

$$y = M-x = M-\frac{M-N}{A+S}S$$

$$= \frac{MA+MS-MS+NS}{A+S}$$

$$= \frac{MA+NS}{A+S},$$

i.e. $$\text{speed of arm} = \frac{MA+NS}{A+S}.$$

When A is fixed, $M = 0$ and speed of arm $= \dfrac{NS}{A+S}$.

When S is fixed, $N = 0$ and speed of arm $= \dfrac{MA}{A+S}$.

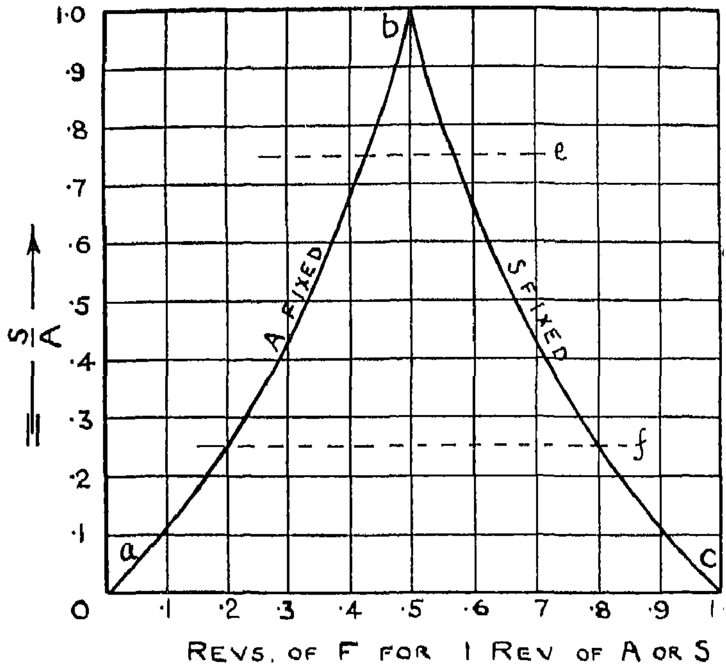


Fig. 87.

Taking a series of values of $\frac{S}{A}$ ranging from 0 to 1, the speed of the arm can be calculated for 1 revolution of A or S depending on which is fixed. These results are shown plotted in Fig. 87, in which values of $\frac{S}{A}$ are plotted vertically against corresponding speeds of the arm for the two cases when either A or S may be fixed. Although values of $\frac{S}{A}$ in the

vicinity of 0 and 1 are outside the limits of practical gears, the curves serve to show the possibilities of the gear.

When A is fixed the curve ab is obtained showing that the gear always acts as a reduction gear with a value always less than $\frac{1}{2}$. When S is fixed the curve bc is obtained, and again the gear acts as a reduction gear with a value lying always between $\frac{1}{2}$ and 1. In both cases the direction of the arm is in the same direction as the driving member. The horizontal dotted lines e and f show approximately the practical limits of the ratio of $\frac{S}{A}$ and of corresponding speeds of the arm.

§ **118. Pre-selective Gear-box.** One form of pre-selective epicyclic gear-box is shown diagrammatically in Fig. 88 and consists of four sun and planet gears in series.

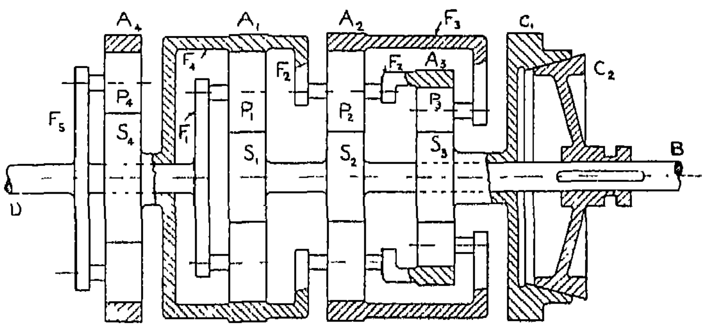


Fig. 88.

The motor shaft B from the engine is keyed to the inner member of the clutch C_2; this shaft is also integral with the sun wheels S_2 and S_1. The outer member of the clutch C_1 is integral with the sun wheel S_3 and this gears through the planet wheel P_3 with the annular wheel A_3. The pin on which P_3 can rotate freely is carried by the arm F_3 which is integral with the annular wheel A_2. The sun wheel S_2 gears through the planet wheel P_2 with the annular wheel A_2. The pin on which P_2 rotates freely is carried by the arms F_2; these arms are integral with annular wheels A_3 and A_1. The sun wheel S_1 gears through the planet wheel P_1 with the annular wheel A_1. The arm F_1 carries a pin on which the planet wheel

P_1 can rotate freely, and this arm is integral with the arm F_5 and both arms form part of the propeller shaft D. The arm F_4 is integral with the annular wheel A_1 and the sun wheel S_4, and the latter gears with the annular wheel A_4 through the planet wheel P_4 and this wheel rotates freely on a pin carried by the arm F_5.

Brake bands (not shown) encircle A_4, A_1, A_2, and C_1 which enable any one of these to be locked in turn. The outer part of the clutch C_1 can only be locked when the inner part C_2 is disengaged from C_1 and this disengagement also applies when the other wheels are locked in turn. The clutch is engaged only on top gear. When any one of the brake bands comes into operation the clutch is automatically disengaged.

First gear. To engage first gear the brake band round A_1 is tightened and A_1 thereby becomes locked and the clutch is disengaged. The sun wheel S_1 is driven by the motor shaft and the drive is from S_1 through P_1 to the arm F_1 and thence to the propeller shaft.

To find the speed of the propeller shaft, let

$$\begin{aligned} \text{speed of motor shaft} &= 1{,}000 \text{ revs. per min.,} \\ \text{teeth on } A_1 &= 82, \\ \text{,, ,, } S_1 &= 26, \\ M_1 &= \text{speed of } A_1, \\ N_1 &= \text{,, ,, } S_1. \end{aligned}$$

From Art. 117 in the sun and planet gear

$$\text{speed of arm } F_1 = \frac{M_1 A_1 + N_1 S_1}{A_1 + S_1}.$$

In this case $M_1 = 0$, $N_1 = 1{,}000$.

Hence

$$\text{speed of arm } F_1 = \frac{1{,}000 \times 26}{82+26} = 240{\cdot}8 \text{ r.p.m.}$$

and speed of propeller shaft for first gear is 240·8 r.p.m.

Second gear. To engage second gear, the brake band round A_2 is tightened, and A_2 is thereby locked, and again the clutch is disengaged. The drive now is from S_2, through the planet wheel P_2 to the arm F_2. Thus F_2, and consequently

A_1, rotates at a definite speed when A_2 is locked. Since S_1 is driven at engine speed, A_1 is driven at a definite speed, then the arm F_1 must rotate at some definite speed.

To find the speed of the propeller shaft, let

$$\text{teeth on } A_2 = 82,$$
$$\text{,, ,, } S_2 = 26,$$
$$\text{speed of } A_2 = M_2,$$
$$\text{,, ,, } S_2 = N_2,$$

then $$\text{speed of arm } F_2 = \frac{M_2 A_2 + N_2 S_2}{A_2 + S_2}.$$

In this case $M_2 = 0$ and $N_2 = 1{,}000$.

Substituting values

$$\text{speed of } F_2 = \frac{1{,}000 \times 26}{82+26} = 240{\cdot}8 \text{ r.p.m.}$$

Considering now the sun and planet gear represented by A_1, P_1, and S_1, the annular wheel A_1 is driven at 240·8 r.p.m. and the sun wheel S_1 at 1,000 r.p.m.

Hence $$\text{speed of arm } F_1 = \frac{M_1 A_1 + N_1 S_1}{A_1 + S_1}.$$

Substituting values

$$\text{speed of } F_1 = \frac{240{\cdot}8 \times 82 + 1{,}000 \times 26}{82+26} = 423{\cdot}5 \text{ r.p.m.}$$

and speed of propeller shaft for second gear is 423·5 r.p.m.

Third gear. Referring to Fig. 88 it will be seen that the sun and planet gears represented by $A_1 P_1 S_1$ and $A_2 P_2 S_2$ respectively are identical with respect to the tooth numbers and that the annular wheel is locked in both cases. If the sun and planet gear represented by $A_3 P_3 S_3$ is a duplicate of the first two sun and planet gears, it is found that the speed of the propeller shaft is then too low, and the expedient of locking the sun wheel instead of the annular wheel is adopted to give an increased speed to the propeller shaft. In the third gear the sun wheel S_3 is locked by clamping the outer part of the clutch C_1, the inner part of the clutch C_2 being disengaged.

To find the speed of the propeller shaft, let

$$\begin{aligned} \text{teeth on } A_3 &= 73, \\ \text{,, ,, } S_3 &= 23, \\ M_3 &= \text{speed of } A_3, \\ N_3 &= \text{ ,, ,, } S_3, \end{aligned}$$

F_2 refer to speed of arm F_2,

F_3 ,, ,, ,, F_3.

Since F_3 is integral with A_2, then $F_3 = M_2$,
and since F_2 ,, ,, A_3, then $F_2 = M_3$.

Considering the sun and planet gears $A_3 P_3 S_3$,

$$\text{speed of arm } F_3 = \frac{M_3 A_3 + N_3 S_3}{A_3 + S_3}.$$

Substituting values and remembering that $N_3 = 0$ since S_3 is locked,

$$F_3 = M_2 = \frac{M_3 \times 73 + 0}{73 + 23} = \frac{M_3 \times 73}{96} = \frac{F_2 \times 73}{96}.$$

Considering next gears $A_2 P_2 S_2$,

$$\begin{aligned} F_2 &= \frac{M_2 A_2 + S_2 A_2}{A_2 + S_2} \\ &= \frac{F_2 \times \frac{73}{96} \times 82 + 26 \times 1{,}000}{82 + 26}, \end{aligned}$$

from which $F_2 = 570$ r.p.m.

Finally, considering gears $A_1 P_1 S_1$,

$$\text{speed of } F_1 = \frac{M_1 A_1 + N_1 S_1}{A_1 + S_1},$$

and since $M_1 = F_2$,

$$\begin{aligned} \text{speed of } F_1 &= \frac{570 \times 82 + 1{,}000 \times 26}{82 + 26} \\ &= 673 \text{ r.p.m.} \end{aligned}$$

Hence speed of propeller shaft on third gear is 673 r.p.m.

Fourth or top gear. For top gear all the brake bands are free and the two parts of the clutch C_1 and C_2 are engaged. Since C_1 is connected to S_3 then both S_2 and S_3 rotate at engine speed of 1,000 r.p.m.

Considering the gears $A_2 P_2 S_2$ and $A_3 P_3 S_3$,

then
$$F_3 = M_2 = \frac{M_3 A_3 + N_3 S_3}{A_3 + S_3},$$

$$M_2 = \frac{M_3 \times 73 + 1{,}000 \times 23}{73 + 23}$$

and
$$96M_2 = 73M_3 + 23{,}000.$$

Also
$$F_2 = M_3 = \frac{M_2 A_2 + N_2 S_2}{A_2 + S_2},$$

$$108M_3 = 86M_2 + 26{,}000,$$

from which $M_3 = M_2 = 1{,}000$ r.p.m.

Similarly, it may be shown that the whole gear rotates as a complete unit at 1,000 r.p.m. Hence speed of propeller shaft on top gear is 1,000 r.p.m.

Reverse gear. For reverse gear the brake band round A_4 is clamped and A_4 is thus locked, the clutch being again disengaged. Let

number of teeth on $A_4 = 91$,

,, ,, ,, $S_4 = 47$.

M_4 = speed of A_4,

N_4 = ,, ,, S_4,

F_5 refer to speed of arm F_5,

F_4 ,, ,, ,, F_4,

F_1 ,, ,, ,, F_1.

Considering the gears $A_1 P_1 S_1$ and $A_4 P_4 S_4$,

$$F_1 = \frac{M_1 A_1 + N_1 S_1}{A_1 + S_1} = \frac{M_1 \times 82 + 1{,}000 \times 26}{82 + 26},$$

$$F_5 = \frac{M_4 A_4 + N_4 S_4}{A_4 + S_4} = \frac{0 + F_4 \times 47}{91 + 47} = \frac{0 + M_1 \times 47}{138}.$$

F_1 and F_5 are integral, hence $F_1 = F_5$,

and
$$\frac{82M_1 + 26{,}000}{108} = \frac{M_1 \times 47}{138},$$

$$M_1 = -577 \text{ r.p.m.}$$

Hence $$F_5 = -\frac{577 \times 47}{138} = -196{\cdot}5 \text{ r.p.m.}$$

Hence speed of propeller shaft on reverse gear is $-196{\cdot}5$ r.p.m.

EXERCISES. VII

1. In a reverted train an arm A carries two concentric separate wheels B and C and a compound wheel DE. B gears with E and C with D. B has 75 teeth, C 30, and D 90. Find the speed and direction of C when B is fixed and the arm A makes 100 revs. per min. clockwise. [*Inst. C. E.*]

2. Make a sketch of the application of an epicyclic train of wheels to any purpose which has come under your notice, and show how to find the relative velocity of the first and last piece in the train. [*Inst. C. E.*]

3. An epicyclic gear consists of a wheel A with 84 internal teeth and a pinion B, and a wheel C of 40 teeth concentric with A, B gears with C and A. The arm which carries the axis of B rotates at 20 revs. per min. If A is fixed, find the speed of C; and if C is fixed, find the speed of A.

4. A pulley-block for lifting heavy weights is constructed as follows: Fixed to the casing so as not to revolve is an annular wheel A of 30 teeth. A second wheel B, of very nearly the same diameter, has one more tooth than A; it revolves loosely on a spindle concentric with A, and is bolted to a pulley 8 in. diameter, round which is a chain by which the weight is lifted. A spur wheel C can turn freely at the end of an arm which is keyed to the spindle, and this wheel is deep enough to gear with both wheels A and B. To the spindle is also fixed a pulley of 12 in. effective diameter, around which is placed the chain for the effort. Find the velocity ratio of haul to lift.

5. Sketch the arrangement by means of which the lathe spindle can be connected to the leading screw of a screw-cutting lathe through a train of wheels.

If the pitch of the leading screw is $\frac{1}{4}$-in. right-hand single thread, find a suitable train of wheels for cutting a right-hand screw thread having 14 threads per in. [*I. Mech. E.*]

6. In the train of wheels shown in Fig. 89 the annular wheel A is fixed and the wheel B is free to rotate about the spindle S. The arm EE is also free to rotate about S and carries planet wheels P, which engage with wheels A and B. The number of teeth on wheels A, B, and P are 120, 90, and 15 respectively.

Find the angles through which the arm EE and the wheels P will move during a complete revolution of wheel B. [*I. Mech. E.*]

7. In a reverted train of wheels, the toothed wheels A, D have a common axis O, but they can move independently. They gear respectively with wheels B, C, which are both fixed to a common spindle S,

whose axis is parallel to the axis A and D. The axis of the spindle S is kept at constant distance from O by means of the arm OS which can rotate about O.

Find expressions in terms of the wheel diameters for the angle through which D will turn: (a) if A is fixed and the arm OS makes a complete revolution; (b) if OS is fixed and the wheel A makes a complete *revolution*.

If the arm OS is 3 in. long and is fixed, find the diameters of the wheels so that the velocity ratio may be 400. [*I. Mech. E.*]

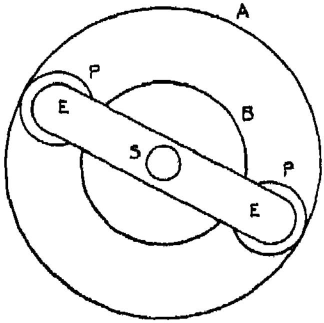


Fig. 89.

8. A speed-reduction gear between two shafts having the same axis consists of an epicyclic train as follows: Wheel A is keyed on the driving shaft. Wheel B gears with A and also with the fixed annular wheel C. Wheels B and D are fixed to a common spindle which is carried by an arm which can rotate about the axis of the wheel A, and the wheel D gears with the annular wheel E, which is keyed to the driven shaft. If A has 20 teeth, B 24, and D 16, and all teeth have the same pitch, find the velocity ratio of the two shafts.

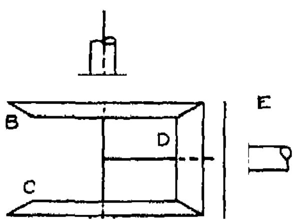


Fig. 90.

9. In the train of bevel wheels shown in Fig. 90, wheels A, B, and C have a common axis, wheels A and B being keyed to the same shaft. The wheel D gears with B and C and can rotate freely about its axis, which can rotate about the common axis of A, B, and C. Wheel E

gears with A. The wheel diameters are A, 8; B and C, 5; D, 3; and E, 4 in. respectively.

If E makes 60 revs. per min. about its fixed axis, find:

(a) the speed of C if the axis of D is fixed;

(b) the speed of D about its own axis if the wheel C is fixed;

(c) the speed of the axis of D about the common axis of A, B, and C if the wheel C is fixed. [*I. Mech. E.*]

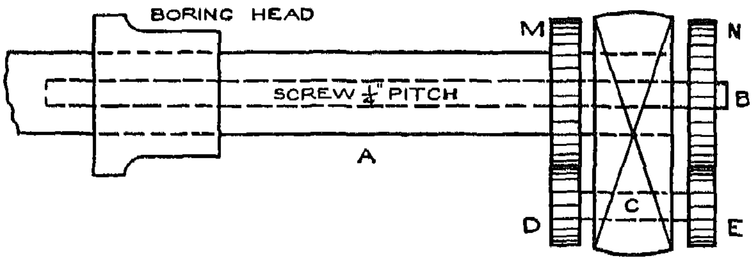


Fig. 91.

10. Fig. 91 shows the feed mechanism of a boring machine. The boring head slides along the boring bar A, its motion along the shaft being produced by a screw B which has $\frac{1}{4}$-in. pitch. A spur wheel M of 64 teeth is keyed to the bar A and a similar wheel N of 64 teeth to the screw. A lay shaft C carries two wheels keyed to it, D and E, which mesh with the wheels M and N. D has 41 teeth and E has 42 teeth. Find the feed, that is the movement of the boring head along the bar per revolution of the bar. [*Inst. C. E.*]

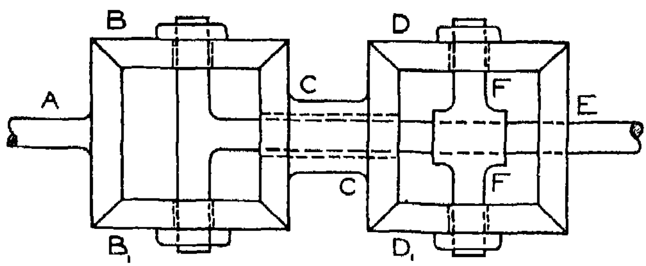


Fig. 92.

11. A compound differential gear, with wheels of uniform size, is shown diagrammatically in Fig. 92. The wheel A is fixed to the driving shaft. Wheels B and B_1 are mounted freely on arms which form part of the driven shaft. Wheels D and D_1 are mounted freely on arms FF, which also can rotate on the driven shaft and which are driven by an external agency. Wheel E is fixed. Wheel A and the

arms FF revolve in the same direction, wheel A at 200 and arms FF at 60 revs. per min. Determine the direction of rotation and speed of the driven shaft. [*Inst. C. E.*]

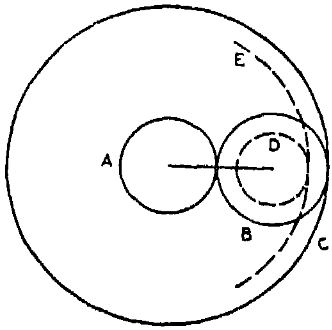


Fig. 93.

12. In an epicyclic gear similar to that shown in Fig. 93, the wheel C is fixed, wheel E is keyed to a machine shaft, and the wheel A to a motor shaft. Wheels B and D rotate together on a pin carried by the arm which rotates about the shaft on which A is fitted. The wheels A, D, and B have 15, 12, and 20 teeth respectively.

If the motor shaft runs at 800 revs. per min., find the speed of the machine shaft.

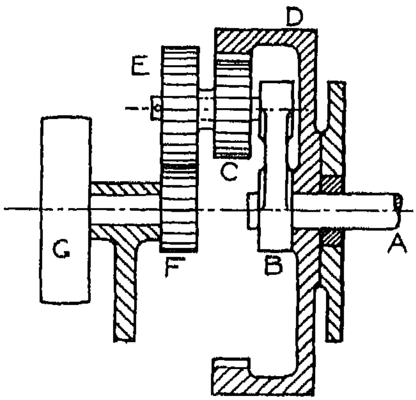


Fig. 94.

13. In the epicyclic reduction gear shown in Fig. 94 a shaft A is driven by an arm B, which is fixed to it. B has a pin fixed to its outer end, and two pinions C, E, which are cast together in one piece, revolve on this pin. C gears with an annular fixed wheel D, and E

gears with a pinion F, which is driven by a belt-pulley G. The numbers of teeth are as follows:

$$D = 80;\ C = 15;\ E = 24;\ F = 18.$$

The pulley G runs at 240 revs. per min.; find the speed of the shaft A. [*Lond. B.Sc.*]

14. Fig. 95 shows the diagram of an epicyclic gearing. A and B are spur wheels having their centres connected by a lever, and each wheel is free to revolve on its axis. C is a wheel with internal teeth concentric with A. Find the numbers of teeth on wheels B and C for the wheel C to revolve 130 times when the wheel A, which has 60 teeth, is kept fixed and the arm from A to B is made to revolve 100 times in the same direction as the wheel C. [*Lond. B.Sc.*]

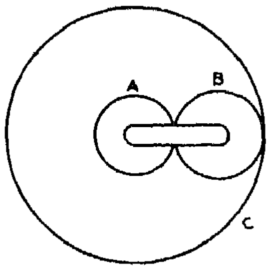


Fig. 95.

15. It is required to connect two parallel shafts by a mechanism which will cause a pointer to indicate the difference of their speeds. Sketch a suitable mechanism. [*Lond. B.Sc.*]

16. The minute hand of a clock is attached to a spindle and the hour hand to a sleeve rotating loosely on the same spindle. Find a suitable train of wheels A, B, C, D to satisfy the requirements, all teeth to have the same pitch. Each wheel must have more than 9 teeth. Make the total number of teeth as small as possible. [*Lond. B.Sc.*]

17. A mechanism for recording the distance travelled by a bicycle is as follows:

There is a fixed annular wheel A of 22 teeth, and another annular wheel B of 23, which revolves loosely on the axis of A.

An arm, driven by the bicycle wheel through gearing not described, also revolves freely on the axis of A and carries on a pin at its extremity two wheels C and D, which are attached to each other. The wheel C, with 19 teeth, meshes with A, and the wheel D, with 20 teeth, meshes with B.

The diameter of the bicycle wheel is 28 in. What must be the velocity ratio between the bicycle wheel and the arm, if B makes one revolution for each mile covered? [*Lond. B.Sc.*]

18. Fig. 96 shows a train of wheels.

C gears with A and B.

D is on the same axis as C.

D gears with E and F with G.

A has 40 teeth; B, 80 internal; C, 20; D, 50; E, 20; F, 40; and G, 90.

The arm makes 10 revs. per min. in a clockwise direction, and the wheel A is at rest.

Find the number of revolutions of B and G respectively per min. Also determine the turning-moment on the shaft carrying the wheel G when the arm is turned with a turning-moment of 10,000 in.-lb. [*Lond. B.Sc.*]

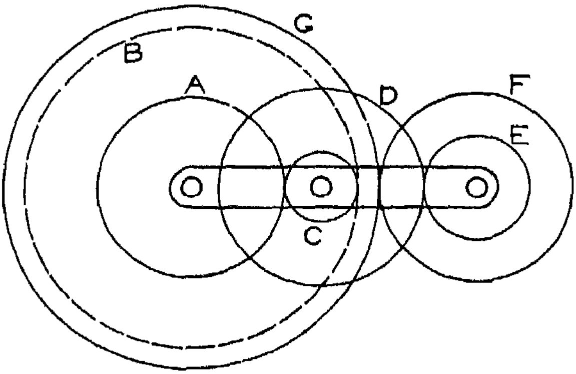


Fig. 96.

19. The train of wheels shown in Fig. 97 is used to transmit motion between two shafts A and B. The wheel fixed to A has 30 teeth and rotates at 500 revs. per min. C gears with A and is rigidly fixed to D, both being free to rotate on B. D gears with F, which is rigidly connected to the pulley E. E and F rotate freely on the shaft B. The wheels C, D, and F have 50, 70, and 90 teeth respectively. By means of a belt drive E is caused to rotate at 80 revs. per min. in a direction opposite to that of A. Determine in magnitude and direction the speed of the shaft B. [*Lond. B.Sc.*]

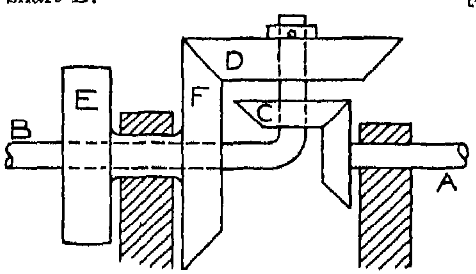
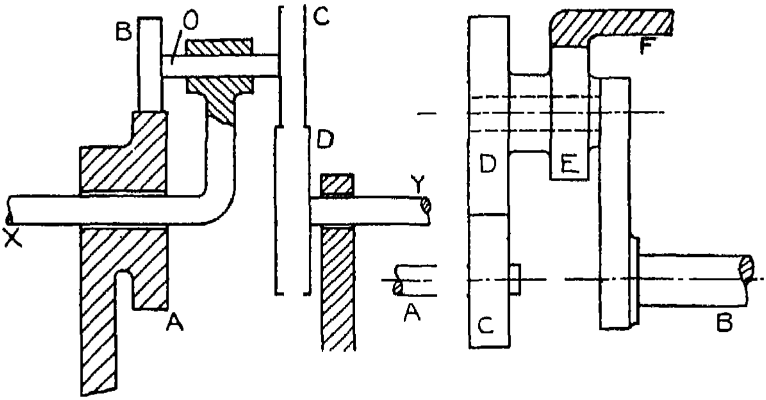


Fig. 97.

20. A reduction gear consists of a train of four spur wheels, Fig. 98 (*a*), of which A is fixed and has 72 teeth, while B and C are secured to a shaft O, carried by an arm rotating with the driving shaft X and gearing with the wheels A and D in the manner shown. Find the direction of motion and number of revolutions of the shaft Y for 1,000 revolutions of the shaft X, if the numbers of teeth on the gear wheels B, C, and D are 12, 13, and 71 respectively. [*Lond. B.Sc.*]

21. If in the gear shown in Fig. 98 (*a*) the wheels A and B have 72 and 12 teeth respectively, find the numbers of teeth for C and D in order that Y shall revolve at the same speed as X but in the opposite direction.



(*a*) Fig. 98. (*b*)

22. In the epicyclic reduction gear shown in Fig. 98 (*b*) the pinion C is keyed on the shaft A. The gears D and E are connected together and ride on a crank arm which is keyed to shaft B. The gear wheel E is in mesh with a fixed annular gear F. The numbers of teeth are:

$$C = 21;\ D = 28;\ E = 14;\ F = 84.$$

If A runs at 100 revs. per min., what is the speed of B?

[*Lond. B.Sc.*]

23. In the gear shown in Fig. 99, A is a fixed wheel of 200 teeth; the wheel B (40 teeth) gears with A and is free to rotate on an arm D. The wheel C (30 teeth), compound with B, gears with the wheel E (20 teeth), which can rotate round the vertical spindle. The wheel F (60 teeth) is compound with E and gears with wheel G (16 teeth). Find the number of revolutions of wheel G for one rotation of the arm D about the vertical spindle.

24. Three spur wheels A, B, and C in gear are mounted on an arm which makes 60 revs. per min. about the axis of A. A is fixed and has 120 teeth; C has 60 teeth. Find the direction and speed of rotation of C.

A pointer whose length is equal to the distance between the axes of A and C is mounted on the axis of C. Show that its free end moves in a straight line with simple harmonic motion. [*Lond. B.Sc.*]

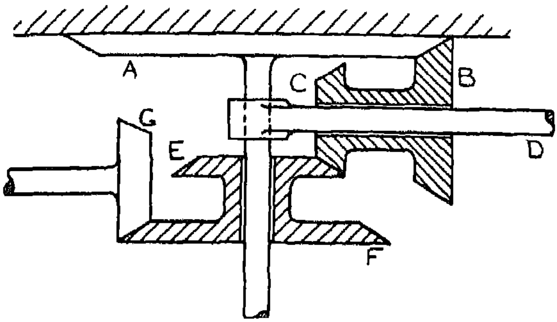


Fig. 99.

25. Fig. 100 shows the arrangement of part of a pre-selective gear-box. Keyed to the motor shaft B are two equal sun wheels S_1 and S_2 having 24 teeth; these gear, through the planet wheels P_1 and P_2, with equal annular wheels A_1 and A_2 having 80 internal teeth. The annular wheel A_1 carries a pin on which P_2 can rotate freely, and P_1 can rotate freely on a pin carried by the arm F keyed to the propeller shaft C. Brake bands (not shown) enable A_1 and A_2 to be locked in turn.

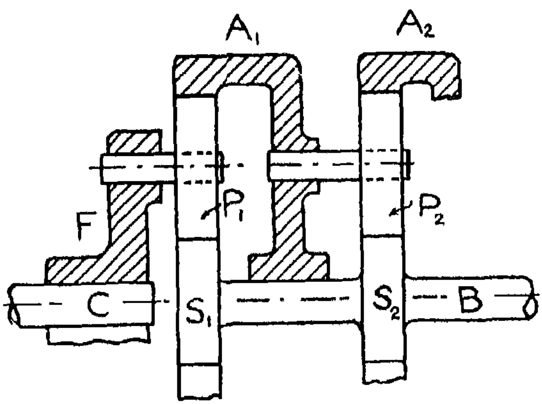


Fig. 100.

For a motor speed of 1,000 revs. per min. find the speed of the propeller shaft—

(i) when A_1 is locked;

(ii) when A_2 is locked. [*Lond. B.Sc.*]

26. In the epicyclic gear shown in Fig. 82 the annular wheel C has the same number of teeth internally and externally and is driven by the pinion D rotating about a fixed centre. The pinion A is driven at constant speed from a motor and D can be driven at varying speeds through friction gearing (not shown) in the same direction as A.

If the speed of A is constant at m revs. per min. and D has extreme speeds of $(m \pm n)$ revs. per min., giving corresponding speeds to E of $\pm a$ revs. per min., show that the number of teeth on A must be equal to the number of teeth on D. [*Lond. B.Sc.*]

27. An epicyclic gear for a hoist block is shown in Fig. 101. The arm E is keyed to the same shaft as the load drum and the wheel A is keyed to a second shaft which carries a chain wheel, the chain being operated by hand. The two shafts have a common axis but can rotate independently. The wheels B and C are compound and rotate together on a pin carried at the end of the arm E. The wheel D has internal teeth and is fixed to the outer casing of the block so as not to rotate.

The wheels A and B have 16 and 36 teeth respectively with a diametral pitch of 4. The wheels C and D have a diametral pitch of 3. Find the numbers of teeth on C and D for the speed of A to be ten times the speed of E, both rotating in the same direction.

[*Lond. B.Sc.*]

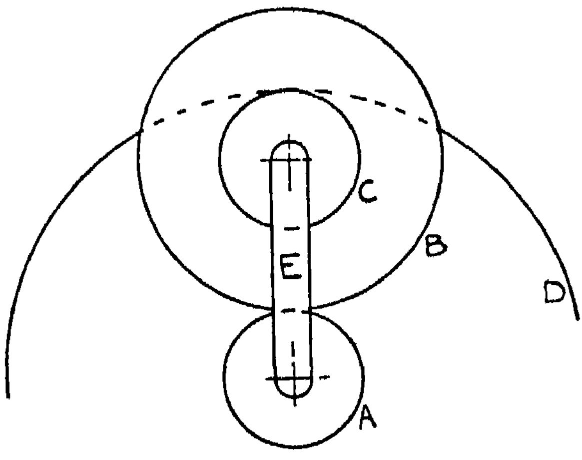


Fig. 101.

28. Part of an epicyclic gear-box is shown in Fig. 86. The gear consists of an annular wheel A concentric with a sun wheel S. The planet wheel P gears with A and S and can rotate freely on a pin carried by the arm F. The wheel A is fixed and the arm F can rotate about the same centre as S but independently of the latter.

If T_1 = number of teeth on S,

T_2 = number of teeth on A,

show that the ratio of speeds S to F is $\dfrac{T_1+T_2}{T_1}$.

If the least number of teeth on any wheel is 15 and $T_1+T_2 = 114$, find the greatest and least speeds of the arm F when the wheel S rotates at 1,000 revs. per min. [*Lond. B.Sc.*]

29. A differential reduction gear is shown in Fig. 102. On the shaft A, driven at a constant speed of 600 revs. per min., are keyed pinions C and C_1; the pinion C_1 gears with the spur wheel D_1, and C gears with an idle wheel E which in turn gears with D. The wheels D and D_1 are compound with the bevel wheels F and F_1 respectively, the latter gearing with the bevel pinions G and G_1 which are mounted on the arm H but free to rotate about their own axes. The arm H is keyed to the slow speed shaft B. The compound wheels DF and D_1F_1 are each free to rotate about the axis of the shaft B. Find the speed and direction of rotation of the shaft B. [*Lond. B.Sc.*]

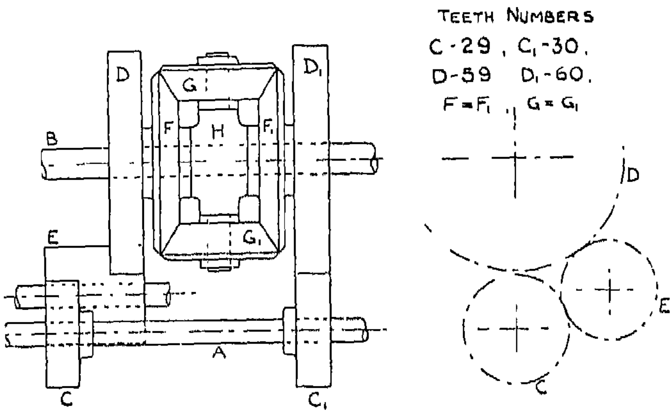


Fig. 102.

30. The epicyclic gear shown in Fig. 103 represents a Port and Starboard Engine Indicator. The two engines rotate in opposite directions and a difference in speed is indicated by a rotation of the pointer P which is mounted on the shaft S. The wheels CD and EF are compound. The arm H which is mounted on the shaft S carries a bevel wheel G which can rotate freely about its own axis.

Find the speed of the pointer P when the Port and Starboard engines rotate at 90 and 100 revs. per min. respectively, and find the number of revs. made by G about its own axis. [*Lond. B.Sc.*]

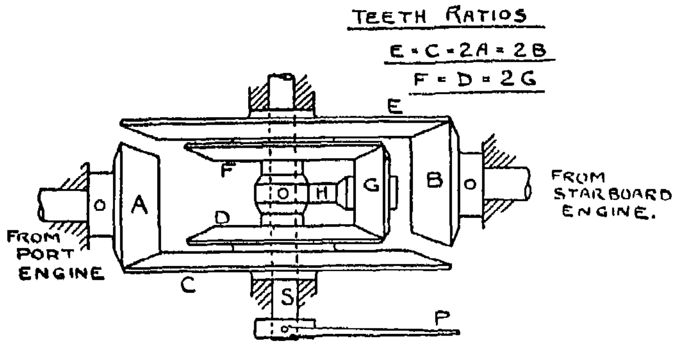


Fig. 103.

31. Fig. 104 shows a synchronizer for two turbines running in the same direction. On the turbine shafts are two pulleys of equal size, driving the pulleys G and D by means of open belts.

The pulley G is keyed to shaft A, upon which is also keyed a gear wheel B. The pulley D is keyed to a sleeve which carries arms C, and these arms carry pins upon which the pinions F can freely rotate. An annular wheel E gears with the pinions F, and these in turn gear with B.

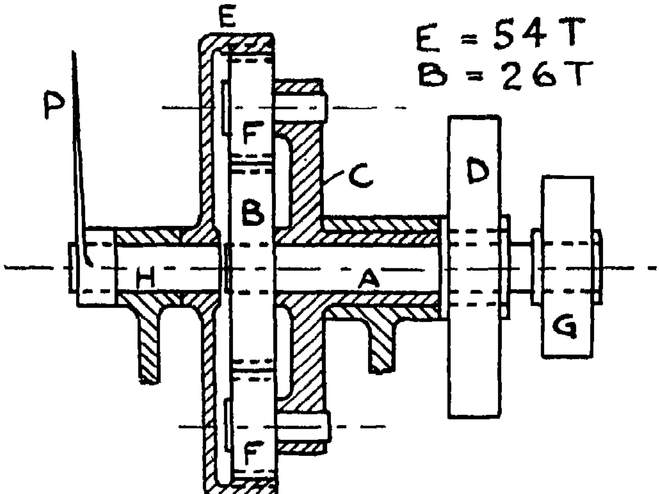


Fig. 104.

The annular wheel E is keyed to a spindle H, to which is fixed the pointer P. When the turbines are running at the same speed the pointer remains stationary. Find the ratio of the diameters of the pulleys D and G. [*Lond. B.Sc.*]

CHAPTER VIII

FRICTION

§ **119. Friction Force.** The resistance experienced when one body slides over another, with which it is in contact, is called the friction force or the force of friction. The friction force always opposes or tends to oppose motion, and for this reason it is desirable in many machines to reduce the friction force to the least possible value. On the other hand, it must be remembered that the successful operation of many machine components depends upon this friction force, and it is desirable in these circumstances to increase the friction force as much as possible. Instances of this latter case occur in belt drives, friction clutches, rolling of wheels on rails or roads, etc.

It is convenient to distinguish between the various kinds of friction by division into two sections. First, there is the friction that occurs between unlubricated surfaces, and secondly, that between lubricated surfaces. These two divisions may be further subdivided depending upon the conditions existing between the two surfaces.

(i) *Unlubricated Surfaces.* The friction between unlubricated surfaces can be split up into two sections:

(*a*) *Dry Friction.* The friction between perfectly clean and dry solid surfaces. This is usually referred to as solid friction.

(*b*) *Rolling Friction.* The friction between solid surfaces separated by balls or rollers.

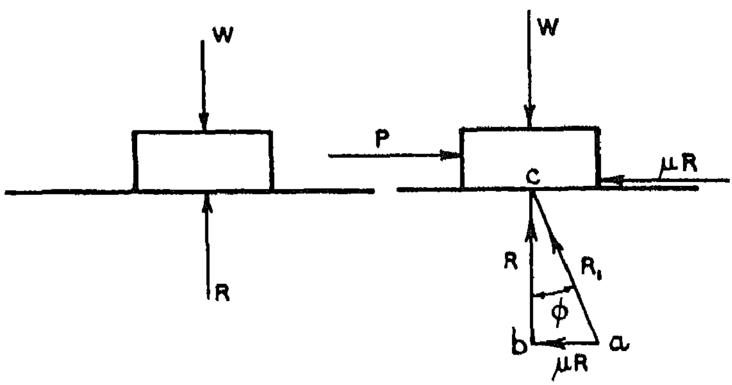
(ii) *Lubricated Surfaces.* The friction between two surfaces between which a lubricant is present may be divided into two sections:

(*c*) *Greasy Friction.* The friction between two solid surfaces between which is an extremely thin layer of oil or grease. This is referred to as Boundary, Greasy, Partial, or Non-viscous friction.

(*d*) *Fluid Friction.* The friction that occurs when the two solid surfaces are separated completely by a film of

lubricant. This is referred to as Viscous or Perfect lubrication.

§ **120. Coefficient of Friction.** When a body is at rest on a horizontal plane and the body is in direct contact with the plane, i.e. there is no fluid or lubricant between the surfaces, the surfaces are said to be dry, and any friction between the surfaces is called solid friction. When the body is at rest, as in Fig. 105 (*a*), the weight of the body acts vertically downwards and the plane exerts a force vertically upwards equal in magnitude to the weight. This force is called the reaction of the plane and is an illustration of Newton's third law of



(*a*) Fig. 105. (*b*)

motion. The body is in equilibrium and the reaction R is equal to W. R is the normal reaction of the plane.

When a horizontal force P is applied to the body as in Fig. 105 (*b*), the resulting motion, if any, will depend upon the value of P. When the force P is gradually increased so that the body is just on the point of slipping, the friction force is just equal to P and in this case it was called the static friction force. If motion takes place and the body slides over the plane with uniform motion, i.e. without acceleration, the friction force is equal to the force P and in general is slightly less than the static friction force. In Fig. 105 (*b*), P is the force applied to the body to cause steady uniform motion. It is found by experiment that the friction force opposing motion

is always proportional to the normal reaction R; in other words, the friction force opposing motion is equal to the normal reaction multiplied by a constant. This constant is called the coefficient of friction and is designated by the Greek letter μ.

$$\text{Friction force} = \mu R.$$

In the case of a horizontal plane R is equal to W, but in the case of an inclined plane R is not equal to W. Referring to Fig. 105 (b), since R is a force acting on the body and μR is a force opposing motion, these may be compounded into a single force. Representing the force μR by a vector ab and the normal reaction by a vector bc, the sum of these vectors is ac, which represents the sum of the forces or the resultant reaction. Call this resultant R_1; then R_1 is inclined to the normal reaction at an angle bca, and $\tan bca = \frac{\mu R}{R} = \mu$. The angle bca is usually designated by ϕ and is called the friction angle or angle of friction. The effect of friction is to displace the reaction through an angle ϕ such that $\tan \phi = \mu$.

When the motion of the body is steady and uniform, the friction force is called sliding friction and in general is rather less than the static friction.

§ **121. Friction between Solid Dry Surfaces.** Experiments carried out by a number of investigators have revealed certain relations between the friction force and characteristics of the body in contact with a plane. For two given materials the friction force depends upon the nature of the materials, such as the hardness and smoothness of the surfaces in contact, and since these are capable of such wide variation, it is exceedingly difficult to estimate the friction force that will exist between two given surfaces. Further, it is difficult to ensure perfect metal to metal contact, and a slight film of air between the surfaces may modify considerably the friction force.

Assuming that the pressure between two surfaces is not excessive and that the speed is not high, there are three so-called laws of friction which, while not strictly true under all circumstances, may be regarded as sufficiently accurate as to

be of value when estimating the effects of friction. The laws are:

(1) The frictional resistance is directly proportional to the normal pressure between the surfaces.

(2) The frictional resistance is independent of the area of contact between the surfaces.

(3) The frictional resistance is independent of the speed for moderate speeds and may increase with decrease of speed.

The first law is practically a definition of the coefficient of friction, i.e. the friction force $= \mu \times$ normal pressure or reaction between two surfaces.

The second law is approximately true, and the advantage of using a large surface compared with a small one is that the heat generated by friction is more readily dissipated. When the heat generated between two dry surfaces cannot be dissipated fast enough, the surfaces ultimately fuse to a certain extent, and they are said to 'seize'.

The third law includes the experimental verification of the fact that the static friction is usually greater than sliding friction.

When the speed between two surfaces is high and when the temperature rises due to the generation of heat, the coefficient of friction is very often less than that at moderate speeds and normal temperature.

§ **122. Rolling Friction.** When a spherical ball or a roller rolls over a flat surface, the contact is theoretically at a point or along a line parallel to the axis of the cylinder, respectively. Practically, however, the ball or roller possesses weight and in the case of ball or roller bearings is usually under pressure. The effect of this pressure is to cause deformation of the surface and also of the ball or roller. This deformation always takes place—even with the hardest of surfaces—although the amount of deformation may be extremely small. The effect of this deformation is to cause an area of contact rather than point or line contact. The amount of deformation depends upon the pressure and the elasticity of the materials in contact.

Considering the case of a roller under pressure in contact with a flat surface, an area of contact is established, and slip occurs towards the extremities of the area of contact with consequent frictional resistance. The laws of friction relating to rolling have been summarized by Professor Goodman and may be briefly stated thus:

1. The coefficient of friction of ball bearings is practically constant at all speeds, but may decrease slightly with increase of speed.
2. The coefficient of friction of ball bearings with flat races decreases with increase of load, but with grooved races the coefficient sometimes increases as the load is increased. The coefficient is much more constant than that of plain lubricated bearings.
3. The coefficient of friction is independent of the temperature of the bearing.
4. The coefficient is practically the same when starting as when running.

The friction of well-designed ball bearings is slightly higher when lubricated than when dry, but the application of a small amount of lubricant is useful in preventing the formation of rust.

§ **123. Greasy Friction.** When two metallic surfaces are apparently in contact, the application of a small amount of lubricant causes a very thin film to be formed on each of the surfaces. This thin film adheres to the surface and is said to be an *adsorbed* film, and it is now recognized that such a film is really in chemical union with the surface. Although the thickness of an adsorbed film is probably of molecular dimensions, the effect is to reduce considerably the friction force between two metallic surfaces compared with the condition when they are dry and free of adsorbed films.

The capacity of a lubricant to form an adsorbed film upon a metallic surface depends upon a property of the lubricant that may be described as 'oiliness'. The 'oiliest' oil is one which most reduces the coefficient of friction. With greasy friction the friction force between two metallic surfaces in

apparent contact is proportional to the normal pressure between the surfaces, and the laws governing greasy friction are, in general, similar to those for solid or dry friction.

§ **124. Viscous Friction.** The frictional resistance between two metallic surfaces apparently in contact can be reduced still further by the introduction of a film of lubricant, of sufficient thickness to separate completely the two surfaces. This film of lubricant must not be confused with the film which exists in boundary or greasy lubrication; this latter is of extreme thinness, probably of molecular dimensions, while the former is comparatively much thicker. The frictional resistance depends upon the thickness of this film, the area of the surfaces in contact, the relative velocity between the surfaces, and the viscosity of the lubricant. When relative motion between two surfaces, separated completely by a lubricant, takes place, an extremely thin film adheres to each of the surfaces and moves with that surface. Resistance to motion is thus the resistance offered by the lubricant itself and not by the so-called contact of the surfaces. This resistance depends upon the 'body' or viscosity of the lubricant and is due to the resistance to sliding or shear of adjacent layers of the lubricant.

If two such surfaces are considered, one at rest and the other having tangential motion, that film of lubricant in contact with and adhering to the moving surface moves with it, while the film in contact with the stationary surface remains at rest. The lubricant between these two films may be considered as consisting of a series of films or layers each moving at a speed proportional to the distance from the surface at rest. The force necessary to produce relative motion between two such surfaces is the force required to cause these adjacent layers to slide over each other.

§ **125. Coefficient of Viscosity.** It has been seen that, when two surfaces are separated completely by a film of lubricant and relative motion between the surfaces takes place, a force is required to shear the lubricant or cause adjacent layers to slide over each other. This resistance of the layers to

sliding is termed internal friction and the lubricant is said to exhibit the property of viscosity.

Maxwell's definition of the coefficient of viscosity is as follows:

The coefficient of viscosity of a fluid is measured by the tangential force on unit area of either of two horizontal planes at unit distance apart, one of which is fixed while the other moves with unit velocity, the space between the surfaces being filled with the viscous substance.

Consider two flat horizontal surfaces between which a fluid is present. Let the lower plate be fixed and the other plate move with a velocity v in a direction parallel to its surface. The layer in contact with the fixed surface is at rest while the layer in contact with the moving surface has a velocity equal to v. The intervening space may be considered as consisting of a series of parallel layers, the velocity of a particular layer depending upon its distance from the fixed surface.

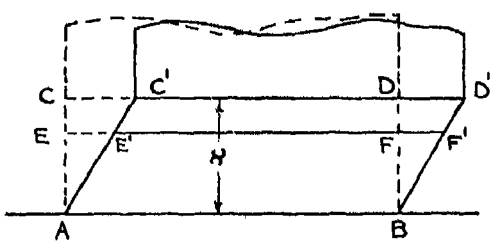


Fig. 106

In Fig. 106, let AB represent the fixed surface and CD the moving surface whose velocity is v. In a given interval of time CD has moved to $C'D'$ and the layer of fluid in contact with CD has accordingly moved to $C'D'$. For any other layer such as EF, its new position is $E'F'$ and EE' represents the displacement.

Let P = force producing motion of CD,

a = area of plate CD,

x = distance between the plates,

f = intensity of shear stress,

$\eta =$ coefficient of viscosity,

$v =$ velocity of CD;

then $f = \dfrac{P}{a}$

and $\dfrac{v}{x} =$ shear strain which is constant over the distance x.

Now $\dfrac{v}{x}$ is the rate of distortion, and the stress required to produce unit rate of distortion is the coefficient of viscosity.

Thus
$$\eta = \frac{f}{\frac{v}{x}}.$$

According to Maxwell's definition the coefficient of viscosity is the force per unit area, i.e. stress, to cause unit velocity when the distance apart is unity. Hence when $v = 1$ and $x = 1$
$$\eta = f.$$
It will be seen that the coefficient of viscosity is thus a stress per second and is measured in dynes per square centimetre per second.

The property of viscosity is possessed by all fluids and a rough generalization can be given that a thick oil is more viscous than a thin one; that is, it offers a greater resistance to flow.

§ **126. Measurement of Coefficient of Viscosity.** Although the coefficient of viscosity is defined by its resistance to shear, it is more usual to adopt other means of measuring the viscosity than that of directly measuring the resistance to shear. An instrument for measuring the viscosity of a fluid is termed a viscometer, and several forms have been devised notably the Ostwald, the Redwood, the Engler, and the Saybolt. Where great accuracy is required, the Ostwald viscometer is usually employed, but for commercial testing the Redwood type is frequently used in this country.

The principle of the Redwood viscometer is as follows: A certain quantity of fluid at a known temperature is allowed to flow through an orifice and the time of flow is observed. The time taken for a given quantity of fluid to flow is known

as the Redwood viscosity number and is expressed in seconds. To convert this viscosity number into absolute units of viscosity (as defined by Maxwell) a conversion formula is required.

In general, the coefficient of viscosity of most lubricating oils decreases with increase of temperature.

§ **127. Journal-bearing Lubrication.** The experiments of Mr. Beauchamp Tower in 1883 for the Institution of Mechanical Engineers were the first important experiments to demonstrate that in viscous friction the two surfaces of a journal and its bearing are completely separated by a film of oil. Tower's experiments were made on a journal 4 inches in diameter, the lower part of the journal being always immersed in oil. The bearing consisted of a half brass resting on the journal. The lubricant was thus applied on the unloaded side of the journal. The half bearing was held in a frame and arrangements were made so that the frame could be loaded with different loads. When the journal was rotated the frame tended to rotate and rotation was prevented by applying small weights at the end of an arm on the frame. Thus the frictional torque could be measured.

The existence of a fluid film under pressure was demonstrated by drilling a series of holes in the bearing and connecting these to a manometer, the pressure at each hole being measured in turn. For a journal rotating in a bearing and separated from it by a film of lubricant the frictional resistance is:

(*a*) proportional to the speed,
(*b*) proportional to the area,
(*c*) proportional to the viscosity of the lubricant,
(*d*) independent of the pressure,
(*e*) independent of the materials in apparent contact.

In general, the coefficient of friction is expressed by

$$\mu = \frac{P}{W},$$

where P = frictional resistance,
W = normal pressure;

and this expression is frequently used to express the equivalent coefficient of friction for a viscous bearing. In this case P represents the frictional resistance at the periphery of the journal or of the bearing and $P \times r$ represents the frictional torque, where $r =$ radius of bearing. Hence μWr represents the frictional torque and the value of μ becomes equal to the frictional torque divided by Wr.

The results obtained by Tower led to a mathematical investigation by Osborne Reynolds, who showed that for a film to be maintained between two surfaces they must be slightly inclined to each other thus forming a wedge of lubricant between the surfaces.

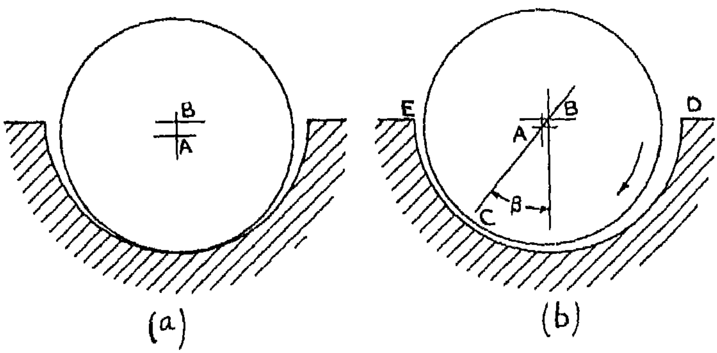


Fig. 107.

§ **128. Eccentricity of a Bearing.** In order that a journal may rotate freely and to ensure the formation of an oil film, the diameter of the bearing must be slightly greater than that of the journal. This difference is shown exaggerated in Fig. 107, in which a half bearing is shown. When the journal is at rest, the lubricant is squeezed out because of the pressure between the shaft and the bearing, and the journal comes into metallic contact with the bearing, the line of contact being vertically underneath the centres A and B of the journal and bearing as shown in Fig. 107 (a). When motion of the journal takes place, the frictional resistance is initially that due to greasy friction. A film of oil adheres to the surface of the journal and is carried round with it until—due to increasing speed—the film becomes of sufficient thickness to lift the

journal. In this manner the pressure of the oil gradually rises. The journal now takes up a new position as shown in Fig. 107 (*b*). The line joining the centres *A* and *B* is no longer vertical but inclined, and the inclination of this line to the vertical is called the attitude of the bearing, denoted by β in the diagram. The distance between *A* and *B* (Fig. 107 (*b*)) is known as the eccentricity of the bearing.

The clearance space between the journal and the bearing thus takes the form of two curved wedges, one on either side of *C*, the point of nearest approach. This clearance space is filled with oil under varying pressure. The pressure increases from zero where the oil enters at the 'on' side at *D* to a maximum at a point somewhere near *C*, the point of nearest approach; the pressure then falls to zero before reaching the 'off' side at *E* where the oil is free to escape. The wedge of oil formed between *D* and *C* satisfies the condition for the formation of an oil film between two surfaces and the pressure of the oil is sufficient to lift the journal and to separate completely the journal and the bearing; in this condition the laws of viscous friction apply.

§ 129. **Effect of Speed on Coefficient of Friction.** When the journal in Fig. 107 is at rest, the two surfaces are more

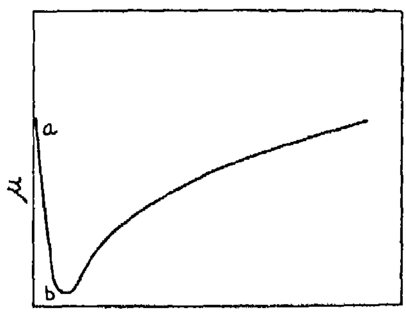


Fig. 108.

or less in direct contact and the starting friction is akin to greasy friction. When rotation begins, the coefficient of friction falls rapidly as the speed increases, since the oil is fed

between the surfaces. This decrease in the coefficient is shown in Fig. 108 in which the variation of the coefficient of friction is shown plotted against the rubbing speed. From *a* to *b* the conditions vary from greasy friction at *a* to viscous friction at *b*, at which stage the complete oil film is formed and viscous friction begins After passing the speed denoted by the point *b*, the resistance of the oil film increases at a greater rate than the speed of rubbing, and consequently the coefficient increases. The minimum value is constant for all pressures, but as the pressure increases, a higher velocity is required to establish the oil film.

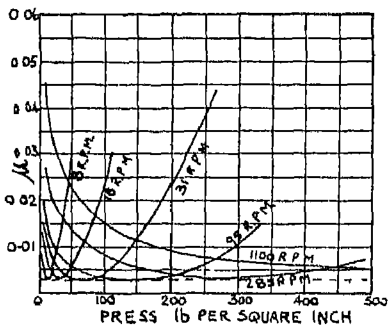


Fig. 109.

§ **130. Effect of Load and Speed.** The variation of the coefficient of friction with the load and speed is shown in Fig. 109. These curves are the results of experiments made by Stribeck on a Sellers self-aligning cast-iron bearing $2\frac{3}{4}$ inches in diameter and 9 inches long. The pressure is plotted horizontally against the coefficient of friction, and a series of curves is shown for different speeds ranging from 8 to 1,100 revolutions per minute. From these curves it is seen that at a speed of 8 r.p.m., the pressure at which viscous friction begins is about 12 lb. per square inch. As the pressure increases, a higher velocity is required to cause the formation of the oil film. Thus at a pressure of about 270 lb. per square inch the speed must be increased to 285 r.p.m. to produce the oil film.

It will be seen from the diagram that the minimum coefficient of friction is the same in each case—about 0·003 as

indicated by the horizontal dotted line. From this it follows that for any particular pressure at which a bearing is loaded, there is some definite velocity to give the minimum value of the coefficient of friction, and that this minimum value is the same for all pressures. This fact is of considerable importance in determining the length of a bearing.

§ **131. Bearing Surfaces.** It has been shown that for perfect film lubrication of a bearing the surfaces of the journal and its bearing are completely separated by a film of the lubricant and that the frictional resistance is that due to resistance to shear the lubricant. Since the rubbing surfaces do not come into direct metallic contact, and the layers of lubricant adjacent to the metallic surfaces are not moving relative to the surfaces, the surfaces theoretically are not subject to wear and the nature of the surfaces can have no influence on the coefficient of friction.

In practice, however, wear does take place even under the most favourable conditions. This is probably due to the failure of the oil film at starting and stopping. When the journal is at rest the metallic surfaces are more or less in contact—any oil film present being in the nature of an adsorbed film and of extremely small dimensions. The conditions at starting and stopping approximate to greasy friction. Consequently, at starting, wear takes place up to the moment when the viscous oil film is formed and at stopping from the cessation of the viscous oil film. Extremely small particles of metal are produced by wear and these float about in the lubricant as it circulates through the bearing. In course of time these particles increase in number and accumulate and tend to cause further wear. In addition, the presence of small particles of solid matter may tend to destroy the formation of the viscous oil film and the working condition may gradually alter to one which is a combination of viscous film and greasy lubrication. To minimize the evil effect of small particles of solid matter an oil filter is frequently fitted.

§ **132. Resistance of Viscous Bearing.** For a journal rotating in a bearing with viscous lubrication, the journal is

separated from the bearing by a film of oil and the frictional resistance is that due to the viscosity of the oil. If the journal is concentric with the bearing, the film of oil is of uniform thickness and an expression for the friction torque may be developed as follows:

Let t = thickness of oil film,
η = coefficient of viscosity,
d = diameter,
n = speed of journal in revs. per min.,
l = length of bearing.

From § 125, $$\eta = \frac{f}{\frac{v}{x}}$$

or $$f = \eta \frac{v}{x},$$

where f = force per unit area or shear stress. In the case of the viscous bearing x is the thickness of the oil film and v is the tangential velocity of the journal.

Hence $$f = \eta \frac{\pi d n}{60 t}$$

and total frictional resistance

$$= \eta . \frac{\pi d n}{60 t} \times \pi d l.$$

Hence frictional torque = frictional resistance $\times \frac{1}{2} d$

$$= \eta . \frac{\pi d n}{60 t} . \pi d l \times \tfrac{1}{2} d$$

$$= \frac{\eta . \pi^2 d^3 l n}{120 t}.$$

This expression for the friction torque only applies for a uniform thickness of oil film.

Horse-power lost in frictional resistance

$$= \frac{\text{frictional torque in lb.-ft.} \times 2\pi n}{33{,}000}.$$

Example 1. A journal 3 in. in diameter runs in a bearing whose length is 6 in. The thickness of the oil film is 0·002 in. and the coefficient of viscosity is 1·6 in C.G.S. units. Find the horse-power lost in friction when the journal rotates at 250 r.p.m.

Coefficient of viscosity $\eta = 1{\cdot}6$ C.G.S. units

$$= \frac{1{\cdot}6 \times 30{\cdot}5^2}{981 \times 453{\cdot}6} \text{ lb. ft. units.}$$

$$= 0{\cdot}00334 \text{ lb. ft. units.}$$

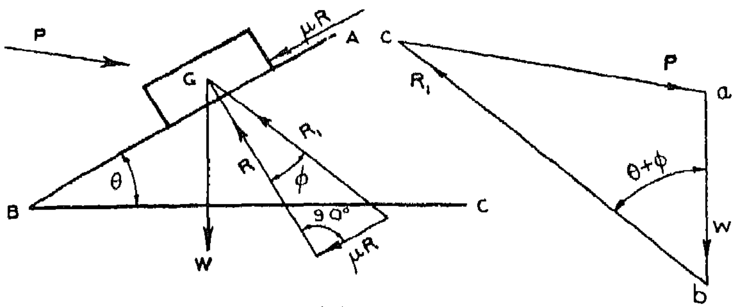
$$\text{Frictional torque} = \frac{\eta \times \pi^2 d^3 ln}{120t}$$

$$= \frac{0{\cdot}00334 \times \pi^2 \times (\tfrac{1}{4})^3 \times \tfrac{1}{2} \times 250}{120 \times \dfrac{0{\cdot}002}{12}} \text{ lb.-ft.}$$

$$= 3{\cdot}22 \text{ lb.-ft.}$$

$$\text{Horse-power lost in friction} = \frac{3{\cdot}22 \times 2\pi \times 250}{33{,}000}$$

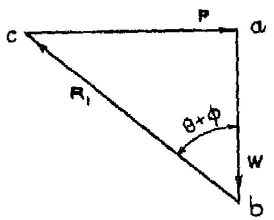
$$= 0{\cdot}153.$$



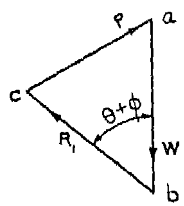
(a) Fig. 110. (b)

§ **133. Motion up Inclined Plane.** In the following examples on the effect of friction it is assumed that the friction force is proportional to the normal pressure between the surfaces. In Fig. 110 (a), let BC represent a horizontal plane and BA an inclined plane, inclined at an angle θ to BC. Let G be the centre of gravity of a weight W. For motion along the inclined plane there are two cases to consider: (1) motion up the plane, (2) motion down the plane. Considering first, motion up the plane, and taking a general case in which the force P, required to cause steady motion, is acting as shown, the value of P may be readily found by considering the forces acting on the body. Let R = normal reaction of the plane, then the friction force which opposes motion and therefore acts down the plane, is μR. The two forces R and μR may be combined by vectors to give a single force R_1, inclined to R

at an angle ϕ, where $\tan\phi = \mu$. The forces acting on the body are: (1) its weight W acting vertically downwards through its centre of gravity, (2) the resultant reaction R_1, and (3) the force P. If the motion is steady, i.e. without acceleration, the body is in equilibrium under the action of these three forces, and when three forces act on a body in equilibrium they must either be parallel or pass through a point. In the given case the forces are not parallel, and hence they must pass through a point, and the value of the forces P and R_1 can be found by drawing a triangle whose sides are parallel to the directions of the three forces. In Fig. 110 (*b*), *ab* is drawn vertical to represent W, *bc* and *ca* are drawn parallel to R_1 and to P respectively. The angle *abc* is equal to $(\theta+\phi)$, and the lengths *bc* and *ca* measured to scale give the magnitudes of the forces R_1 and P.



(*a*)



(*b*)

Fig. 111.

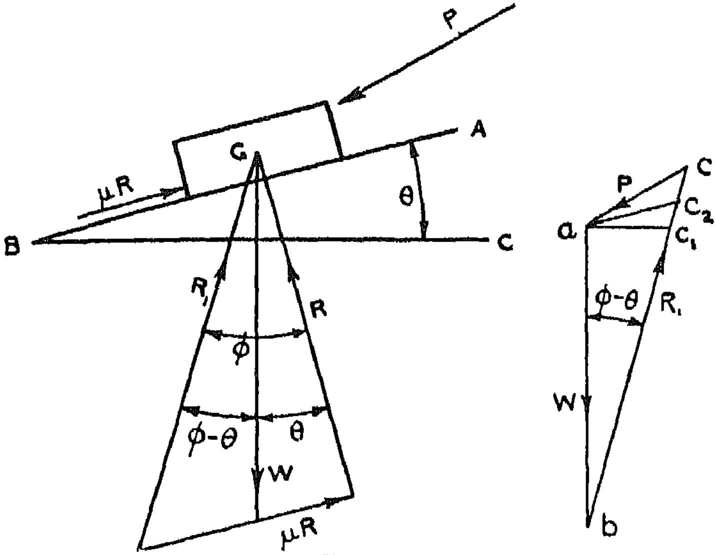
There are two cases which are of practical importance: (1) when P is horizontal, (2) when P is parallel to the plane. In each case the method of procedure is the same as that outlined above. In Fig. 111 (*a*) the triangle *abc* is the triangle of forces for the case when P is horizontal. The vector *ca* is drawn horizontal and it is readily seen that the angle *abc* is $(\theta+\phi)$:

$$\therefore\ P = W\tan(\theta+\phi).$$

When P is parallel to the plane, the triangle of forces is *abc*, Fig. 111 (*b*), in which *ca* is drawn parallel to the plane. On reference to Figs. 110 (*b*) and 111 (*a*) and (*b*), it will be seen that the resultant reaction R_1 has the same direction in each case, but that the magnitude is different.

§ **134. Motion down Inclined Plane.** The force P will be required to assist or prevent motion down the plane

according to whether the angle θ is less than or greater than the angle of friction. Taking the case in which the angle of the plane is less than that of friction, the magnitude of P is found in a similar manner to that already described. In Fig. 112 (a), R is the normal reaction, and since friction opposes the motion, the friction force μR acts up the plane.



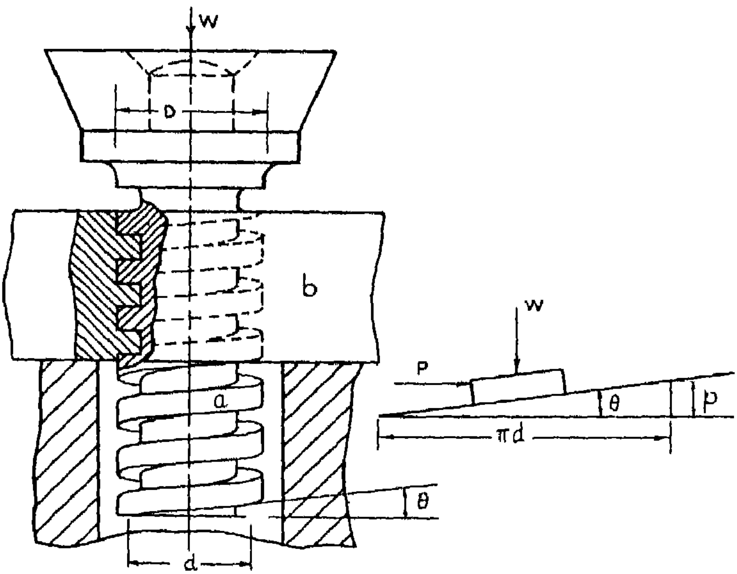
(a) Fig. 112. (b)

The forces R and μR are combined to give the resultant reaction R_1 inclined at an angle ϕ to R. In the general case when P is inclined to the plane, the triangle of forces is shown in Fig. 112 (b) at abc. When P is horizontal the triangle becomes abc_1, the vector ac_1 being drawn horizontal. In this case $P = W \tan(\phi - \theta)$.

When P is parallel to the plane the triangle of forces is abc_2, in which ac_2 represents the value of P necessary to cause motion down the plane.

§ **135. Friction of Square-threaded Screw.** The method applicable for finding the horizontal force required to move a body up or down an inclined plane may be utilized for finding the force required to raise a load by means of a square-threaded screw. A screw thread may be regarded as a square

ribbon of metal wrapped round a cylinder in the form of a helix. In Fig. 113 (*a*), let *a* represent a square-threaded screw which engages with the nut *b*. The top of the screw may be enlarged suitably to carry a load *W*. In the first case, *W* will be assumed to rotate with the screw. The bottom surface of the screw thread is in contact with the upper surface of the thread in the nut, and the development of this



(*a*) Fig. 113. (*b*)

surface is an inclined plane. Neglecting the weight of the screw itself in comparison with *W*, the problem reduces to that of finding the horizontal effort *P* required to move a load *W* up an inclined plane whose inclination with the horizontal is θ, as in Fig. 113 (*b*).

Let d = mean diameter of screw,

p = pitch,

$$\tan\theta = \frac{p}{\pi d}.$$

Let P = force applied at mean radius.

For motion up the plane the triangle of forces is similar to

Fig. 111 (a), in which R_1 is inclined at an angle $(\theta+\phi)$ to W. As in § 133, if μ is the coefficient of friction,

$$\begin{aligned} P &= W\tan(\theta+\phi) \\ &= W\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} \\ &= W.\frac{\dfrac{p}{\pi d}+\mu}{1-\dfrac{p}{\pi d}.\mu} \\ &= W.\frac{p+\mu\pi d}{\pi d-\mu p}. \end{aligned}$$

When friction is neglected, the value of P thus found is the least possible, and the friction angle ϕ becomes zero.

Let P_0 = value of P when friction is neglected, then $P_0 = W\tan(\theta+0) = W\tan\theta$.

The efficiency of the screw is the ratio of P_0 to P.

$$\text{Efficiency} = \frac{P_0}{P} = \frac{W\tan\theta}{W\tan(\theta+\phi)} = \frac{\tan\theta}{\tan(\theta+\phi)}.$$

When raising a load by means of a screw, as in Fig. 113 (a), the effort is usually applied at the end of a lever or spanner; this effort will, of course, be less than that required at mean radius.

Let Q = force required at end of a lever,

l = length of lever from centre of screw.

Then $$Q\times l = P\times\tfrac{1}{2}d,$$

$$\therefore\ Q = \frac{Pd}{2l}.$$

Torque required to turn the screw $= Ql = P\times\frac{1}{2}d$.

When the load does not rotate with the screw, the top of the screw carrying the load does not rotate with the screw and hence it must bear on a collar. Let D = mean diameter of collar and μ_1 = the coefficient of friction between the collar and the bearing surface.

Friction force $= \mu_1 W$.

Friction torque $= \mu_1 W.\frac{1}{2}D$, if the friction force is assumed

to act at mean radius. For practical problems this assumption is usually sufficiently accurate.

$\therefore$ Total torque required to raise load $= P.\frac{1}{2}d+\mu_1 W.\frac{1}{2}D.$

For motion down the plane and with the load rotating with the screw, the triangle of forces is similar to abc_1, Fig. 112 (*b*).

Let P_1 = force required to lower load,

$$P_1 = W\tan(\phi-\theta).$$

The value of the effort at the end of a lever is found in a manner similar to that just described.

EXAMPLE 2. A screw jack has a square-threaded screw of 3 in. mean diameter. The angle of inclination of the threads is 3° and the coefficient of friction is 0·06. It is operated by a handle 18 in. long. What pull must be exerted at the end of this handle: (*a*) to raise, (*b*) to lower, a load of 5 tons? [*Lond. B.Sc.*]

(*a*) Assuming the load to rotate with the screw, the effort at mean radius to raise the load is $P = W\tan(\theta+\phi)$

$$= 5\times 2{,}240\,\frac{\tan 3^\circ+\mu}{1-\tan 3^\circ.\mu} = 11{,}200\,\frac{0\cdot 05241+0\cdot 06}{1-0\cdot 05241\times 0\cdot 06}$$

$$= 1{,}263 \text{ lb.}$$

This effort is applied at $1\frac{1}{2}$ in. radius,

$$\therefore \text{ effort at 18 in. radius} = 1{,}263\times\frac{1\frac{1}{2}}{18} = 105\cdot 2 \text{ lb.}$$

(*b*) Effort at mean radius to lower load $= W\tan(\phi-\theta)$

$$= 5\times 2{,}240\,\frac{0\cdot 06-0\cdot 05241}{1+0\cdot 06\times 0\cdot 05241} = 85 \text{ lb.}$$

$$\therefore \text{ Effort at 18 in. radius} = 85\times\frac{1\frac{1}{2}}{18} = 7\cdot 08 \text{ lb.}$$

§ **136. Friction of Vee Thread.** When the screw and nut have Vee threads, such as the Whitworth screw thread, the friction force is proportional to the normal pressure on the sides of the thread. In Fig. 114 a section of the nut is shown in which the angle of the Vee is 2α. The normal pressure on the slant surfaces is represented by R. This may be resolved into horizontal and vertical components, and the sum of the

vertical components on all the slant surfaces must be equal to W. If R is the total normal pressure,

then $$W = R\cos\alpha,$$

or $$R = \frac{W}{\cos\alpha}.$$

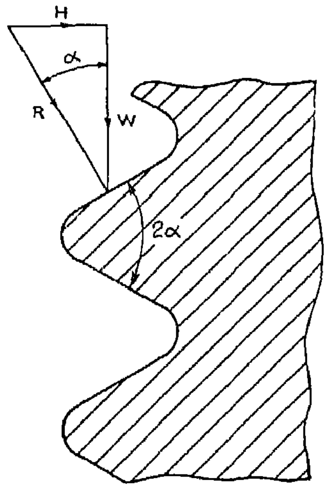


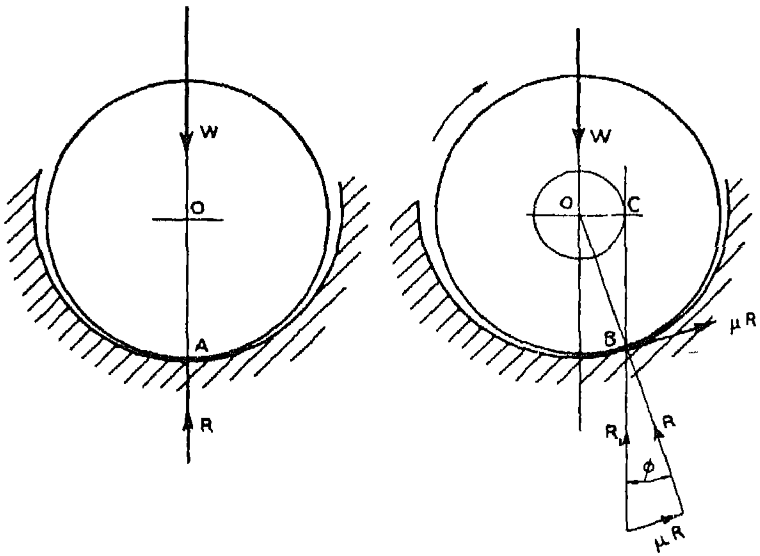
Fig. 114.

The normal pressure is $\frac{W}{\cos\alpha}$ and the results obtained for square threads may be utilized for Vee threads on the substitution of $\frac{\mu}{\cos\alpha}$ for μ, since the friction force is proportional to the normal pressure. This is equivalent to substituting $\frac{\tan\phi}{\cos\alpha}$ for $\tan\phi$ in the formulae for square-threaded screws.

For Whitworth screws $2\alpha = 55°$, and for the American Sellar's thread $2\alpha = 60°$.

§ 137. **Friction at a Journal.** In considering the friction at a journal care must be exercised to distinguish between viscous friction and greasy friction. The case of viscous friction has already been dealt with and in this article the case of greasy friction will only be considered in which the fric-

tional resistance is assumed proportional to the normal pressure. The following theory is only approximate but serves as a useful guide in the estimation of the power lost in a journal with greasy lubrication. The friction force at a journal opposes motion, and since the friction force must act on the outer surface of the journal, the resistance experienced is in the form of a torque. When a shaft is at rest in its bearing,



(a) Fig. 115. (b)

as in Fig. 115 (a), the weight of the shaft acts through its centre of gravity and the reaction of the bearing must be vertically upwards and in line with W. The point of contact A is known as the point or seat of pressure. The clearance between the bearing and the journal is shown exaggerated for clearness of the diagram. When a torque is applied to the shaft such that steady motion takes place, the torque required to produce this steady motion is equal to the friction torque. It is found that the seat of pressure creeps up the bearing to a point B, Fig. 115 (b), in a direction opposite to that of motion.

Let R = normal reaction at the seat of pressure B. Since the two surfaces in contact at B must have a common normal,

this normal must pass through O, the centre of the shaft. The friction force opposing motion is μR and this may be combined with R to give a resultant reaction R_1 inclined at an angle ϕ to R. The forces acting on the shaft are W vertically downwards and R_1, hence for equilibrium R_1 must act vertically upwards and is equal in magnitude to W. The two forces W and R_1, being parallel and not in the same line, together form a couple which must be equal and opposite to the couple or torque producing motion. The couple due to W and R_1 is the friction couple and in magnitude is equal to the value of W multiplied by the perpendicular distance between W and R_1. Let OC be drawn from O perpendicular to R_1.

Friction torque or couple $= W.OC$.

But $OC = r \sin \phi$, where $r =$ radius of shaft.

$\therefore$ Friction couple $= Wr \sin \phi$.

If a circle is drawn with OC as radius, this circle is known as the friction circle of the journal. Since the friction circle is of very small radius compared with the journal, $\sin \phi$ is practically equal to $\tan \phi$, and the friction couple may be written with sufficient approximation as $Wr\mu$. The effect of friction may be regarded as being equivalent to displacing the reaction from the centre of the bearing a distance equal to $r \sin \phi$, or in other words, to displacing the reaction such that it is tangential to the friction circle whose radius is

$$r \sin \phi.$$

§ **138. Horse-power lost in Friction at a Bearing.** It has been seen that the friction torque $= Wr \sin \phi$ or approximately $Wr\mu$. The work done per minute in overcoming friction is the friction torque multiplied by the angle turned through in radians per minute.

Let $N =$ revs. per min.

$$\text{Radians per minute} = 2\pi N.$$

$$\text{Work done per minute} = Wr\mu . 2\pi N.$$

$$\text{Horse-power lost in friction} = \frac{Wr\mu . 2\pi N}{33,000}$$

In the above expression for horse-power W must be in pounds and r in feet.

§ **139. Friction and Slider Crank Chain.** In Fig. 116, let ABC represent a slider crank chain or ordinary engine mechanism. Let P be the effective thrust on the crosshead. Neglecting for the moment the effect of friction, the force P induces a thrust Q in the connecting-rod, and neglecting the inertia effects, the crosshead is in equilibrium due to forces P, Q, and R at A. The magnitude of Q can be found by drawing a triangle to represent the forces P, Q, and R, the

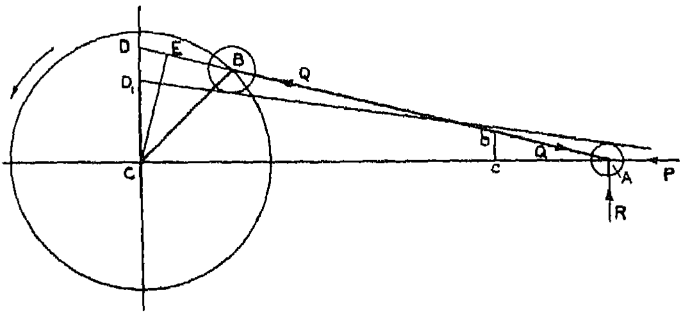


Fig. 116.

sides of the triangle being parallel to the forces P, Q, and R respectively. Such a triangle is represented by Abc, and bA to scale is the thrust Q. The turning-moment on the crank-shaft is Q multiplied by its perpendicular distance from C. Let CE be drawn perpendicular to AB produced.

Turning-moment on crank-shaft $= Q \times CE$.

If AB produced cuts the vertical through C in D, the triangles CDE and Abc are similar, and

$$\frac{CD}{CE} = \frac{Ab}{Ac} \quad \text{or} \quad CD \times Ac \qquad CE \times Ab.$$

$$\therefore\ CD \times P = CE \times Q.$$

But $CE \times Q$ is the turning-moment on the crank, hence $P \times CD$ is the turning-moment. AB is the line of thrust for the connecting-rod.

Taking friction into account at the crosshead pin and crank pin,

let r_1 = radius of pin at A,

r_2 = radius of pin at B,

μ_1 = coefficient of friction at A,

μ_2 = coefficient of friction at B.

Approximately, $r_1 \mu_1$ = radius of circle at A,

$r_2 \mu_2$ = radius of circle at B;

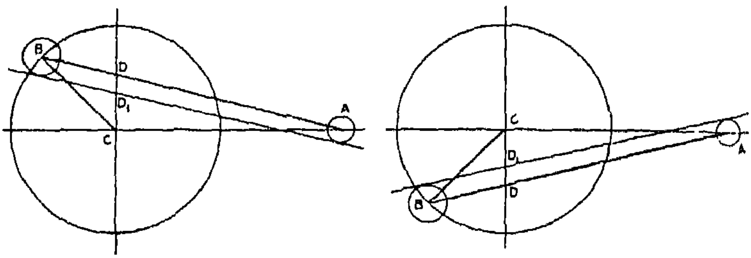
or more exactly, the radii are $r_1 \sin \phi_1$ and $r_2 \sin \phi_2$ respectively. The line of thrust, or, as it is more generally called, the friction axis, is tangential to these friction circles. For the given configuration there are four possible tangents and it remains to decide which one shall be drawn.

Assuming the crank to be rotating in a counter-clockwise direction, for the given configuration the angle BAC is increasing and, relative to the pin at A, the connecting-rod may be regarded as swinging round in a clockwise direction. Relative to the connecting-rod, therefore, the pin at A is rotating in a counter-clockwise direction, and as the seat of pressure creeps up the bearing in a direction opposite to that of rotation of the pin, the tangent at A is on the top side of the friction circle. At B, the angle CBA is decreasing and the crank relative to the connecting-rod may be regarded as swinging in a counter-clockwise direction. The seat of pressure at B creeps round the bearing in a direction opposite to this, and the tangent is at the bottom of the friction circle at B. The friction axis is thus the common tangent to the two friction circles, tangential to the top of the circle at A and to the bottom of the circle at B. Let this common tangent cut the vertical through C in D_1. The turning-moment on the crank is now $P \times CD_1$.

An alternative method of deciding which of the four tangents should be drawn is to remember that the effect of friction is to reduce the turning-moment on the crank, and hence that tangent should be drawn which gives the least intercept CD_1.

Two other positions of the crank are drawn in Figs. 117 (a)

and (*b*) and the corresponding friction axes shown tangential to the friction circles. The effective pressure at the crosshead P multiplied by the intercept CD_1 gives, in each case, the turning-moment on the crank-shaft. The student should have no difficulty in fixing the tangent for a crank position in the remaining quadrant.



(*a*) Fig. 117. (*b*)

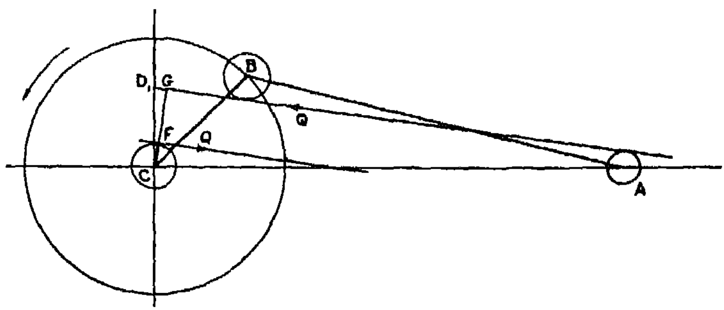


Fig. 118.

§ **140. Dead Angle of Slider Crank Chain.** Considering the friction at the crosshead and crank pins, it has been seen that the thrust is tangential to the friction circles at these pins. In Fig. 118, in which the crank is rotating in a counter-clockwise direction, the friction axis cuts the vertical through C in D_1. The reaction at the crank-shaft bearing is parallel to this and tangential to the friction circle at the crank-shaft. Since the crank-shaft is rotating in a counter-clockwise direction, the seat of pressure creeps round the bearing in a direction opposite to this, and the tangent is thus on the top

side. The turning-moment is now $Q \times FG$, where FG is drawn perpendicular to the friction axis.

When the crank is near the inner dead centre it is possible to find a position in which the reaction Q and the friction axis Q are in the same straight line, as in Fig. 119. The position of the crank is shown at B_1 and a similar position on the other side of the dead centre position is shown at B_2.

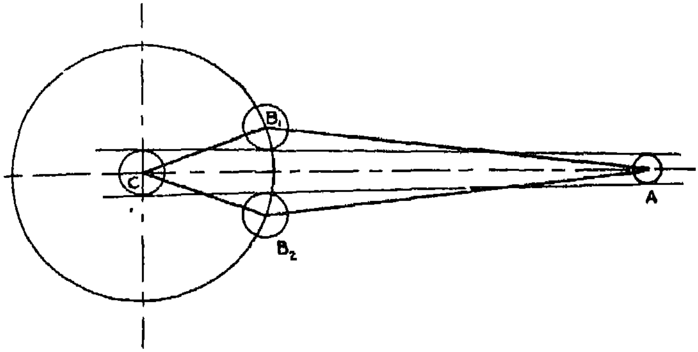


Fig. 119.

The angle B_1CB_2 is known as the dead angle. The friction circles are shown greatly exaggerated for the sake of clearness. There is a corresponding dead angle at the outer dead centre which can readily be found by the method indicated.

§ 141. **Friction of Pivot and Collar Bearings.** When a vertical shaft is subjected to a vertical load or a horizontal shaft to an axial load, the thrust is usually taken by a flat pivot or collar bearing. A pivot is of necessity at the end of a shaft, whereas a collar may be at any position along the shaft, as in the case of collars on propeller shafts of ships. Pivots and collars may be either flat or conical, and these will be dealt with in turn. The thrust is along the axis of the shaft, and the problem to be considered is that of estimating the horse-power lost in friction.

Before the power lost in friction can be estimated the assumptions that may be made must be carefully considered. When a bearing is new it is quite reasonable to expect that

the pressure is uniform over the surface, but since points at different radii are moving at different speeds, it is unlikely that the coefficient of friction is constant for all radii. This point is, perhaps, of greater consequence for flat pivots, in which the velocity at the centre is zero, than for collars. When motion takes place, the rate of wear will be greater at points where the velocity is greater, i.e. greater wear will be experienced near the outer diameter than round about the centre. This variation in the rate of wearing alters the distribution of uniform pressure and the pressure will tend to increase from the outer diameter to the centre. This alteration of pressure again alters the rate of wearing; this effect may be to cause ultimately an even distribution of pressure; but it does not necessarily follow that the pivot again becomes flat; it may become slightly convex or concave, depending upon the materials of the shaft and its bearing. It is thus seen that, once the newness of the bearing has worn off, it is exceedingly difficult to estimate how the bearing may wear and how the coefficient of friction may vary. One of two assumptions is often used in practice, and these are useful in making allowances for friction power, though it is highly probable that neither is strictly correct. The assumptions are that:

(1) the intensity of pressure is constant, or

(2) the wear is uniform.

Taking each of these assumptions in turn, expressions will be found for the friction moment and horse-power lost for pivot, collar, and conical bearings.

(1) *Uniform Pressure.* In Fig. 120 a shaft a with a collar b is subject to an axial thrust P.

Assuming the coefficient of friction constant for all radii, the total friction force is μP.

Let the radii of the shaft and collar be R_2 and R_1 respectively and p the uniform intensity of pressure.

$$p \times \text{area of surface of contact} = P,$$

$$\text{or, } p \times \pi (R_1^2 - R_2^2) = P.$$

Consider an elemental ring of radius r and of width δr.

$$\text{Area of elemental ring} = 2\pi r \times \delta r.$$

$$\text{Pressure on ring} = 2\pi r . \delta r \times p.$$

$$\text{Friction force on ring} = 2\pi r . \delta r . p \times \mu.$$

$$\text{Moment of this force} = 2\pi r . \delta r . p\mu \times r.$$

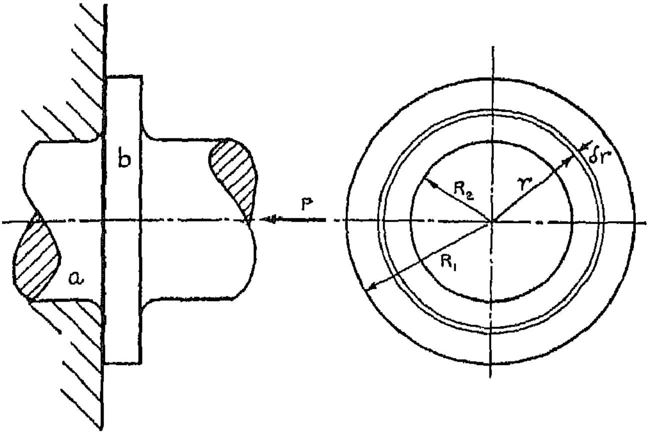


Fig. 120.

$$\therefore \text{ Total moment of friction} = \int_{R_2}^{R_1} 2\pi r^2 dr . p\mu$$

$$= 2\pi \frac{\left[r^3\right]_{R_2}^{R_1}}{3} p\mu = \mu p . 2\pi . \frac{(R_1^3 - R_2^3)}{3}$$

$$= \mu P \times \frac{2}{3} \frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}.$$

Let T = total moment of friction or friction torque,

$$T = \mu P \times \frac{2}{3} \frac{R_1^3 - R_2^3}{R_1^2 - R_2^2}.$$

Horse-power lost in friction $= \dfrac{2\pi N T}{33{,}000 \times 12}$, in which N is the number of revolutions per minute. T is expressed in pound-inch units since the radii are usually measured in inches.

For a flat pivot the investigation is similar, with the

exception that $R_2 = 0$ and R_1 corresponds to the external radius. Putting $R_2 = 0$ in the expression for T,

$$T = \mu P \times \tfrac{2}{3} R_1.$$

The effective radius at which the load can be considered as acting for the purpose of calculating the friction moment is, in the case of a flat pivot with uniform pressure, two-thirds of the external radius.

(2) *Uniform Wear.* Again considering the case of a flat collar (Fig. 120), if the wear is assumed uniform, then the product of the intensity of pressure and velocity may be assumed constant.

$$p \times v = \text{constant},$$

$$\therefore\ p \times r = \text{constant}.$$

Let $p \times r = k$.

Using the same notation as before:

Pressure on the element $= p \times 2\pi r.\delta r = 2\pi k.\delta r$.

Total pressure $= \int_{R_2}^{R_1} 2\pi r.dr.p = \int_{R_2}^{R_1} 2\pi k.dr$,

or
$$P = 2\pi(R_1 - R_2).k,$$

$$\therefore\ k = \frac{P}{2\pi(R_1 - R_2)}.$$

Friction force on element $= 2\pi k.\delta r.\mu$.

Moment of friction force on element $= 2\pi k.\delta r.\mu \times r$.

Total moment of friction $= \int_{R_2}^{R_1} 2\pi k \mu r.dr$

$$= 2\pi k \mu . \frac{R_1^2 - R_2^2}{2}.$$

Substituting the value of k,

$$T = 2\pi\mu . \frac{P}{2\pi(R_1 - R_2)} . \frac{R_1^2 - R_2^2}{2} = \mu P \times \frac{R_1 + R_2}{2}.$$

For a flat pivot, $R_2 = 0$ and R_1 corresponds to the external radius. Putting $R_2 = 0$ in the expression for T,

$$T = \mu P \times \frac{R_1}{2}.$$

The effective radius at which the load can be considered as

acting for the purpose of calculating the friction moment is, in the case of a flat pivot with uniform wear, one half the external radius.

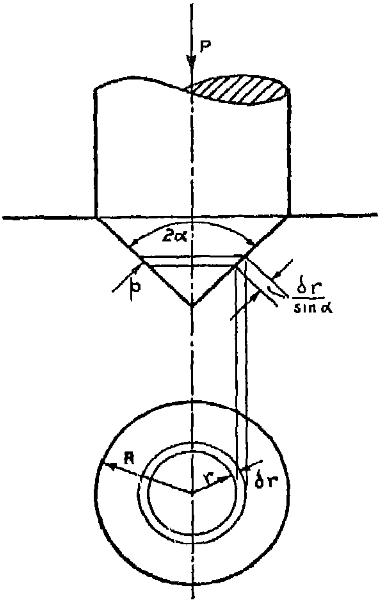


Fig. 121.

§ **142. Friction of Conical Bearings.** In the conical bearing shown in Fig. 121, let P be the axial thrust, 2α the angle of the cone.

(1) *Uniform Pressure.* Consider an elemental ring of radius r and whose plan width is δr. The actual width of the sloping face of the cone is $\dfrac{\delta r}{\sin\alpha}$.

Let p = pressure per square inch normal to the cone surface.

Pressure on elemental ring $= p \times 2\pi r \,.\, \dfrac{\delta r}{\sin\alpha}$.

Vertical component of this

$$= p \times 2\pi r \frac{\delta r}{\sin\alpha} \times \sin\alpha = p \,.\, 2\pi r \,.\, \delta r.$$

$\therefore$ Total vertical pressure $= \int_0^R p \,.\, 2\pi r \,.\, dr = p \,.\, 2\pi \,.\, \frac{1}{2}R^2,$

or $$P = p \,.\, \pi R^2.$$

The intensity of pressure normal to the surface of the cone is thus independent of the angle of the cone.

Friction force on elemental ring $= p \,.\, 2\pi r \,.\, \dfrac{\delta r}{\sin\alpha} \,.\, \mu.$

Moment of this friction force $= p \,.\, 2\pi r \,.\, \dfrac{\delta r}{\sin\alpha} \,.\, \mu r.$

Total moment of friction $= \displaystyle\int_0^R p \,.\, 2\pi r^2 \frac{\delta r}{\sin\alpha} \,.\, \mu,$

or $$T = \mu p \,.\, \frac{2\pi R^3}{3\sin\alpha} = \mu P \times \frac{2}{3} \frac{R}{\sin\alpha}.$$

If the cone is truncated so that the lesser radius is R_2 and the outer radius is R_1, it may similarly be shown that in this case
$$T = \frac{\mu P}{\sin\alpha} \times \frac{2}{3} \frac{R_1{}^3 - R_2{}^3}{R_1{}^2 - R_2{}^2}.$$

(2) *Uniform Wear.* Again assume $p \times r = k.$

Area of elementary ring $= 2\pi r \dfrac{\delta r}{\sin\alpha}$

Pressure on elemental ring $= 2\pi r \dfrac{\delta r}{\sin\alpha} \,.\, p = 2\pi k \,.\, \dfrac{\delta r}{\sin\alpha}.$

Vertical component of this pressure $= 2\pi k \,.\, \dfrac{\delta r}{\sin\alpha} \,.\, \sin\alpha.$

$\therefore$ Total vertical pressure $= \int_0^R 2\pi k \,.\, dr,$

or $$P = 2\pi R k,$$

$$\therefore\ k = \frac{P}{2\pi R}.$$

Friction force on element $= 2\pi k \dfrac{\delta r}{\sin\alpha} \,.\, \mu.$

Moment of friction force $= 2\pi k \, . \dfrac{\delta r}{\sin\alpha} \, . \mu \times r.$

Total moment of friction $= \displaystyle\int_0^R 2\pi k r \frac{dr}{\sin\alpha} \, . \mu$

or $$T = 2\pi\mu k \frac{R^2}{2\sin\alpha} = \mu P \, . \frac{R}{2\sin\alpha}.$$

If the cone is truncated, and $R_2 =$ lesser radius, $R_1 =$ outer radius,
$$T = \mu P \, . \frac{(R_1 + R_2)}{2\sin\alpha}.$$

In all the above cases the horse-power lost in friction is
$$\frac{2\pi NT}{33{,}000 \times 12},$$
T being in inch-pound units.

§ **143. Thrust Bearings.** For a shaft carrying a heavy axial thrust the provision of a single collar may be insufficient, as the bearing pressure would be excessive. To reduce the bearing pressure it is usual to fit a number of collars, the number of collars being determined by the allowable pressure per square inch and the total load. The limiting pressure is about 60 lb. per sq. in. Beyond this pressure too much heat is generated and there is a risk of the bearing 'seizing'.

It has long been established that to maintain a film of lubricant between two surfaces having relative motion, the surfaces should be slightly inclined to each other. This action takes place in an ordinary journal bearing when a slight difference in diameter of the journal and its bearing is allowed. For flat surfaces in contact, one surface should have a slight degree of freedom so that it may become slightly inclined and thus allow a freer entrance for the lubricant. This apparently simple condition is utilized in the Michell thrust bearing, in which only one collar is used. The collar rotates and the bearing consists of a pad or series of pads arranged around the collar in such a manner that they are free to assume a slight inclination to the collar. The pads or bearing blocks are mounted in housings which prevent rotation with the collar,

but which allow the pads to take up a position in which they are slightly inclined to the collar. By this means the length of the bearing may be considerably reduced and the allowable pressure considerably increased. The number of pads may be four to six, depending upon the total thrust, and the allowable pressure may be increased to about 500 lb. per sq. in. The pads are arranged in segments so as to get the maximum bearing area.

EXERCISES. VIII

1. Describe the Michell thrust block, and explain the principle of its action. [*I. Mech. E.*]

2. A load W lb. stands on an inclined plane α to the horizontal. If the coefficient of friction is μ, show that the horizontal force necessary to move the load up the plane is $\frac{\mu+\tan\alpha}{1-\mu\tan\alpha}.W$.

The mean diameter of the square-threaded screw of a screw-jack is $2\frac{1}{2}$ in. and the pitch is $\frac{1}{2}$ in. If the coefficient of friction is 0·08, find the torque necessary to (*a*) raise, (*b*) lower, a load of 3 tons resting on the screw-jack. What is the efficiency of this machine when raising the load? [*I. Mech. E.*]

3. A thrust bearing has four collars of external diameter 22 in., internal diameter 12 in.

If the thrust is 7 tons, estimate the horse-power absorbed in friction when the shaft makes 80 revs. per min. and the coefficient of friction is 0·15, assuming (*a*) uniform pressure, (*b*) uniform wear. [*I. Mech. E.*]

4. Discuss the assumptions usually made in estimating the horse-power absorbed by friction in a footstep bearing.

The load on a vertical shaft, 9 in. diameter, is 18 tons. The shaft rests on a footstep bearing and has a flat end. Assuming a coefficient of friction of 0·06, estimate the horse-power lost in friction when the shaft is running at 100 revs. per min. Assume uniform pressure. [*I. Mech. E.*]

5. A shaft, 3 in. diameter, is supported on bearings 10 ft. apart, and rotates at 100 revs. per min. A pulley, 4 ft. diameter, is keyed to the shaft at 2 ft. from one bearing. Assuming the pull on the pulley due to a belt is 2,500 lb., and that $\mu = 0·08$, determine the horse-power lost in friction at each bearing.

6. A square-threaded screw, $1\frac{1}{2}$ in. in external diameter, 4 threads per in., is used to close a valve, 5 in. in diameter, against a pressure of 100 lb. per sq. in. If $\mu = 0·125$, find the efficiency of the screw and the torque required to turn the handle.

7. Find the horse-power lost in friction at the bearing of a vertical shaft resting on a flat footstep. The load may be assumed to be evenly distributed over the whole surface.

Load	= 5·0 tons
Diameter of shaft	= 8 in.
Revolutions per minute	= 90
Coefficient of friction	= 0·05.

8. Find the force required to pull a mass weighing 2 tons up an inclined plane.

Angle of plane to horizontal	= 20°
Angle between line of action of force and the surface of the plane—the force acts away from the plane	= 30°
Coefficient of friction	= 0·4.

9. A vertical shaft, 2 in. diameter, carries a load of 2,000 lb., and rotates on a flat pivot. Calculate the horse-power wasted in friction when the shaft rotates at 300 revs. per min., assuming (*a*) uniform pressure, (*b*) uniform wear. Take $\mu = 0{\cdot}08$.

10. A conical bearing supports a vertical shaft, 9 in. diameter, which is subject to a load of $3\frac{1}{2}$ tons. The angle of the cone is 120°, and the coefficient of friction is 0·025. Assuming that the pressure is uniform, find the horse-power lost in friction when the speed is 140 revs. per min.

11. The thrust of a propeller shaft in a marine engine is taken up by a number of collars solid with the shaft, which is 12 in. in diameter. The total thrust on the shaft is 18 tons, and the speed is 75 revs. per min. Taking the coefficient of friction to be constant and equal to 0·05, and assuming the intensity of pressure to be constant and equal to 50 lb. per sq. in., find (*a*) the external diameter of the collars, (*b*) the number of collars, required. The horse-power lost in friction is not to exceed 20.

12. The thrust on the propeller shaft of a marine engine is taken by 8 collars, 26 in. in external diameter, the diameter of the shaft between the collars being 17 in. The thrust-pressure is 60 lb. per sq. in., and may be assumed uniform; the coefficient of friction is 0·04, and the speed of the shaft is 90 revs. per min.

Find the horse-power absorbed by friction of the thrust bearing.

13. A collar bearing supports a vertical load of 13 tons. There are 7 collars solid with the shaft, which is 14 in. in diameter; the external diameter of the collars is $25\frac{1}{2}$ in., and the speed of the shaft 85 revs. per min. Assuming uniform wear and $\mu = 0{\cdot}08$, find the horse-power lost in friction.

14. A column weighing 2 tons slides vertically between guides and can be raised or lowered by a horizontal force applied to a wedge on which the column rests. If the angle of the inclination of the wedge

is $\tan^{-1}\frac{1}{8}$, and if the coefficient of friction between all moving surfaces is 0·2, what force is necessary (*a*) to raise, (*b*) to lower, the column? [*Lond. B.Sc.*]

15. A collar thrust bearing has an outer diameter of 9 in. and an inner diameter of 6 in. It rotates at 120 revs. per min., and carries an end thrust of 4 tons. If the coefficient of friction is 0·02, and the wear is assumed uniform, what is the horse-power lost in friction? [*Lond. B.Sc.*]

16. In a flat plate clutch with multiple plates, the number of frictional surfaces is 45. If the inside and outside diameters of the circles over which contact is made are 6 in. and 8 in. respectively, determine the maximum horse-power which can be transmitted at 1,800 revs. per min. when the plates are engaged with a force of 130 lb. Take $\mu = 0{\cdot}15$. [*Lond. B.Sc.*]

17. What is meant by the expression 'Friction Circle'? Deduce an expression for the radius of a friction circle in terms of the radius of the journal and the angle of friction.

For any given crank position in a direct-acting steam engine, show how the turning-moment on the crank-shaft may be found when an allowance is made for the friction at the crosshead pin, crank pin, and crank-shaft bearings. Assume that the effective pressure on the crosshead is known. [*Lond. B.Sc.*]

18. The crank of a direct-acting steam engine is 12 in. long. The crank-pin and crank-shaft journals are 8 in. in diameter. Neglecting the obliquity of the connecting-rod, find the angle through which the crank must turn from a dead centre before there can be any effective torque on the crank-shaft due to the steam pressure on the piston if the coefficient of journal friction is 0·06.

If the diameter of the piston of this engine is 16 in., length of connecting-rod 5 ft., diameter of crosshead pin 4 in., find the effective torque on the crank-shaft due to an effective steam pressure on the piston of 80 lb. per sq. in., when the crank is at right angles to the line of stroke. Allow for friction at guide bars, crosshead pin, and crank pin. $\mu = 0{\cdot}06$. [*Lond. B.Sc.*]

19. What force should be applied at the end of a spanner of 24 in. effective length in tightening up a bolt 2 in. in diameter to resist an axial force of 4,000 lb.? The bolt has a square thread of mean diameter 1·762 in., and the pitch is 0·4 in. The mean diameter of the bolt-head is 2·8 in., and the coefficient of friction is 0·125. [*Lond. B.Sc.*]

20. In a 1-in. diameter Whitworth bolt the mean diameter may be taken as 0·8 in., the pitch $\frac{1}{8}$ in., and the angle of Vee 55°. The bolt is tightened by screwing a nut, whose mean diameter of bearing surface is $1\frac{1}{2}$ in. If μ for the nut and bolt is 0·1, and for the nut and the

bearing surface 0·2, find the force required at the end of a spanner 14 in. long when the load on the bolt is 1,200 lb.

21. A horizontal spur wheel of 80 teeth acts as a nut for a vertical square-threaded screw of 3 in. mean diameter and $\frac{3}{4}$ in. pitch, and raises a load of 4,000 lb. The spur wheel is geared to a pinion of 16 teeth, the efficiency of spur wheel and pinion being 90 per cent. The thrust due to the screw on the spur wheel is taken up by a collar bearing, 4 in. inside diameter and 8 in. outside diameter. The coefficient of friction for the screw and for the collar bearing is 0·15. Find the torque required on the pinion shaft to raise the load. [*Lond. B.Sc.*]

CHAPTER IX

BELTS AND ROPES

§ 144. **Belts.** A continuous belt may be used to transmit rotary motion between two shafts, upon which are mounted pulleys. The distance apart of the shafts is, in general, greater than that suitable for toothed gearing. When the power transmitted is small and the speed is not excessive, the belt moves with the rims of the pulleys without slipping; in practice, however, the power transmitted is usually of such a magnitude that a certain amount of slip takes place, and for this reason the velocity ratio between the shafts is not strictly determinate.

A continuous belt is made by fastening the two ends of a length of belting, the fastening being made by one of several methods. Belting is usually made of leather, though cotton, rubber, and even paper belts are used. Leather belting is made by joining together strips of leather cut from tanned hides. These strips do not usually exceed 5 ft. in length, and a piece of belting, say 30 ft. long, may contain six or seven strips of leather cemented and sewn together in such a manner as to give the appearance of a single length of 30 ft. Single belts are belts composed of a single thickness of leather; double belts are made of two thicknesses of leather sewn together.

The working tension of single belts is usually taken as about 80 lb. per in. of width, although the variation may be from 70 to 100 lb., according to the quality of leather used. Double belts have almost double the strength of single belts, but not quite. The working tension is more usually expressed in lb. per inch of width rather than in lb. per square inch, since the thickness of single belts is approximately constant, except in thin belts, which are made of 'split' leather.

§ 145. **Velocity Ratio between Two Shafts.** The velocity ratio between two shafts connected by a belt on pulleys depends upon the radii of the respective pulleys. In parallel shafts the belt may be *open* or *crossed*. In an open belt

drive the two shafts rotate in the same direction, whereas in a crossed belt drive they rotate in opposite directions. In Fig. 122 (*a*) an open belt is diagrammatically shown, and in (*b*) a crossed belt is shown, in which each portion of the belt as it leaves a pulley is given a half turn so that the same side

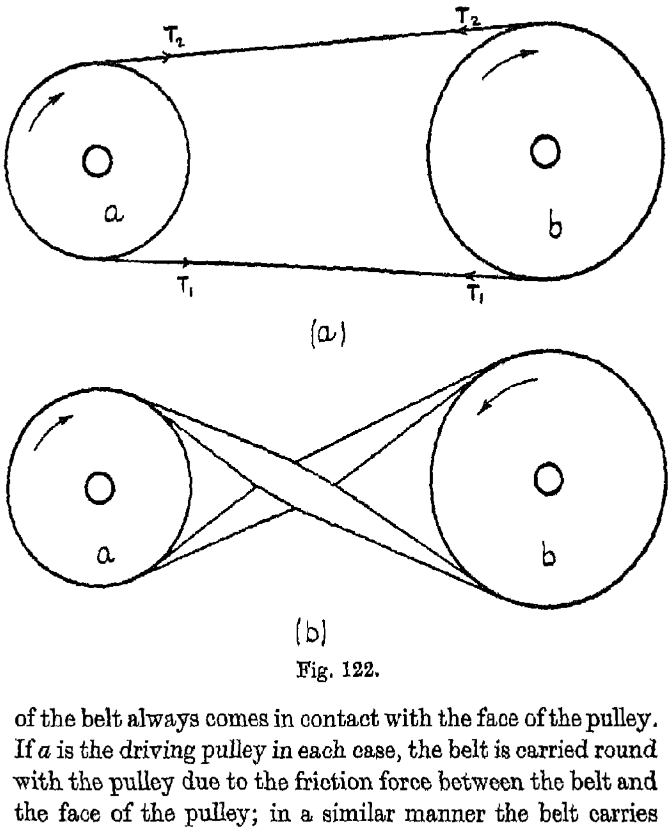


Fig. 122.

of the belt always comes in contact with the face of the pulley. If *a* is the driving pulley in each case, the belt is carried round with the pulley due to the friction force between the belt and the face of the pulley; in a similar manner the belt carries the driven pulley round with it, thus causing motion of pulley *b*. The two pulleys are assumed to be in the same plane, and in order to keep the belt on the pulleys, it is the usual practice to give the pulleys a slight *swell*. The swelling of the pulleys is usually sufficient to keep the belt on, as a belt always tends to ride to the highest point, which is the centre of the face of

the pulley. The lateral stiffness of the belt prevents it from slipping off, although a sudden load coming on the belt may cause it to slip off, and thus prevent damage.

In Figs. 122 (*a*) and (*b*), let *a* and *b* be the driver and driven pulleys respectively.

Let ω_1 = angular speed of driver,

ω_2 = angular speed of follower,

r_1 = radius of driving pulley,

r_2 = radius of following pulley,

N_1 = revolutions per minute of driver,

N_2 = revolutions per minute of follower.

Assuming no slip between the belt and the pulleys and that the thickness of the belt is small, the peripheral speed of the two pulleys must be the same.

$$\text{Velocity of the belt} = \omega_1 r_1 = \omega_2 r_2,$$

$$\therefore \frac{\omega_2}{\omega_1} = \frac{N_2}{N_1} = \frac{r_1}{r_2}.$$

§ **146. Effect of Belt Thickness.** When a belt passes over a pulley, the inner surface of the belt is compressed and the outer surface is stretched, but the neutral surface, which is midway between the outer and inner surfaces, remains unaltered in length. The velocity of a point on the inner surface of the belt as it moves round with the pulley is less than that at the neutral surface.

Let t = thickness of belt,

v = velocity of neutral surface.

Assuming no slip,

$$v = \omega_1(r_1 + \tfrac{1}{2}t) = \omega_2(r_2 + \tfrac{1}{2}t),$$

$$\therefore \frac{\omega_2}{\omega_1} = \frac{N_2}{N_1} = \frac{r_1 + \frac{1}{2}t}{r_2 + \frac{1}{2}t}.$$

The effective radius of each pulley is thus greater than the nominal radius by $\frac{1}{2}t$.

§ **147. Effect of Slip on Velocity Ratio.** If a sudden load comes upon the driven pulley, the friction force between the belt and each of the pulley faces may be insufficient to over-

come this increased resistance and the belt will slip. The slip may occur between the belt and the driver or between the belt and the follower, or between the belt and both pulleys. If the increased load is sudden, the belt may come off one of the pulleys. Assuming the belt to remain on the pulleys and that slip occurs,

let s_1 = percentage slip between driver and belt,

s_2 = percentage slip between belt and follower, and neglecting for the moment the effect of belt thickness,

peripheral speed of driver $= \omega_1 r_1$,

$$\text{speed of belt} = \omega_1 r_1 \times \frac{100-s_1}{100},$$

$$\text{peripheral speed of follower} = \omega_2 r_2 = \frac{100-s_2}{100} \times \text{speed of belt}.$$

$$\therefore\ \omega_2 r_2 = \frac{100-s_2}{100} \times \omega_1 r_1 \times \frac{100-s_1}{100}$$

or
$$\frac{N_2}{N_1} = \frac{\omega_2}{\omega_1} = \frac{100-s_1}{100} \cdot \frac{100-s_2}{100} \cdot \frac{r_1}{r_2}.$$

If S = total percentage slip between the driver and follower,

$$\frac{N_2}{N_1} = \frac{\omega_2}{\omega_1} = \frac{100-S}{100} \cdot \frac{r_1}{r_2}.$$

Taking the thickness of belt into account, the effective radius of each pulley is the nominal radius + half belt thickness,

$$\frac{N_2}{N_1} = \frac{\omega_2}{\omega_1} = \frac{100-S}{100} \cdot \frac{r_1+\frac{1}{2}t}{r_2+\frac{1}{2}t}.$$

EXAMPLE 1. A shaft running at 90 revs. per min. is to drive a parallel shaft at 150 revs. per min. The pulley on the driving shaft is 30 in. in diameter. Find the diameter of the pulley on the driven shaft:

(*a*) neglecting belt thickness;

(*b*) taking account of belt thickness which is $\frac{1}{4}$ in.;

(*c*) assuming in the latter case a total slip of 5 per cent.

(*a*) $\frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}, \quad \therefore\ r_2 = \frac{\omega_1}{\omega_2} \cdot r_1 = \frac{N_1}{N_2} r_1 = \frac{90}{150} \times \frac{30}{2}.$

$$r_2 = 9, \quad \therefore\ d_2 = 18 \text{ in.}$$

(b) $\dfrac{N_2}{N_1} = \dfrac{r_1+\frac{1}{2}t}{r_2+\frac{1}{2}t}, \quad \therefore\ r_2+\dfrac{1}{8} = \dfrac{90}{150}\times 15\tfrac{1}{8}.$

$$r_2 = 8{\cdot}95, \quad \therefore\ d_2 = 17{\cdot}9 \text{ in.}$$

(c) $\dfrac{N_2}{N_1} = \dfrac{100-s}{100}\cdot\dfrac{r_1+\frac{1}{2}t}{r_2+\frac{1}{2}t}, \quad \therefore\ r_2+\dfrac{1}{8} = \dfrac{100}{95}\times\dfrac{90}{150}\times 15\tfrac{1}{8}.$

$$r_2 = 8{\cdot}496, \quad \therefore\ d_2 = 16{\cdot}992 \text{ in.}$$

§ **148. Creep of Belts.** The power transmitted by a belt depends upon the friction force available between the belt and the pulley with which it is in contact. This friction force depends upon the pressure between the belt and the pulley, hence to increase the power transmitted it is required to increase the pressure between the belt and the pulley. This is effectively accomplished by giving the belt an initial tension, before power is transmitted. In other words, when a belt is new and is placed so as to connect two pulleys, the belt is given an initial tightening. Due to the nature of leather, this initial tightening after a few weeks gives rise to a permanent stretch of the belt, and it has to be re-tightened. The initial tension in a belt is sometimes maintained automatically by the use of a jockey pulley.

When a belt transmits power, apart from the initial tension, the belt is subject to driving tensions. Referring to Fig. 122 (*a*) in which *a* is the driving pulley, when motion is taking place, the belt is drawn by *a* on to the underside of the pulley, giving a tension in the belt, say T_1. This tension is constant over the length of belt on the underside of the two pulleys, hence T_1 is urging the pulley *b* round. The belt, being continuous, must move on to pulley *b* at approximately the same rate as it moves off, and the belt is thus pulled off pulley *a* with a tension, say T_2. This tension remains constant for the length of belt on the upper side of the two pulleys. These two tensions T_1 and T_2 are different in magnitude, the difference being the effective force causing either the motion of the belt with pulley *a* or the motion of pulley *b* with the belt. T_1 is known as the tight side tension and T_2 as the slack side tension. That part of the belt leaving the follower and approaching the driver is known as the tight

side, and that part leaving the driver and approaching the follower as the slack side.

A belt, being somewhat elastic, follows more or less the elastic laws. Consider a foot length of belt when unstressed in any way. On the tight side, in which the tension is T_1, the original foot length of this belt will have a length rather more than 1 foot. Let $(1+x_1)$ feet be the new length. On the slack side, tension T_2, let the length of this original foot be $(1+x_2)$ feet. x_1 is greater than x_2. A length of belt $(1+x_1)$ feet approaching the pulley a will leave with a length $(1+x_2)$ feet. Thus, less length of belt leaves the pulley than approaches it. In a similar manner a greater length of belt leaves pulley b than approaches it, and the belt is said to creep, i.e. the belt as it moves over pulley b increases in length slightly, and pulley b rotates at a slightly slower speed than that theoretically calculated. The net effect of creep is to reduce slightly the speed of the driven pulley.

§ 149. Length of Crossed Belt. In practice the length of a belt connecting two pulleys in position, whether open or crossed, is readily found by passing a piece of string around the pulleys and measuring the length required. As, however, allowance must be made for initial tension and for the joint, some experience is necessary in judging the exact length of belt to be cut. It is sometimes necessary to calculate the length of belt required, and the methods for both open and crossed belts will be given.

In Fig. 123, let A and B be the centres of the two pulleys connected by a crossed belt. Let r_1 and r_2 be the respective radii of the pulleys, 2α the angle subtended by the crossed portions of the belt, and d the distance apart of the centres. The angles KAD and MBE are each equal to α. The angle of contact on the smaller pulley is the external angle GAD, or twice angle CAD.

Let l = length of belt,

$$\begin{aligned} l &= 2[\text{arc}\, CD + DE + \text{arc}\, EF] \\ &= 2[r_1(\tfrac{1}{2}\pi+\alpha) + AJ + r_2(\tfrac{1}{2}\pi+\alpha)] \\ &= 2[(r_1+r_2)(\tfrac{1}{2}\pi+\alpha) + d\cos\alpha]. \end{aligned}$$

The angle α is found by drawing AJ parallel to DE and producing BE to J; then $\sin\alpha = \dfrac{BJ}{AB} = \dfrac{r_1+r_2}{d}$.

The length of the belt is constant if (r_1+r_2) is constant, since α depends upon (r_1+r_2); also for a belt connecting two stepped pulleys, the length of belt must, from practical considerations, be constant, and one condition to be satisfied in

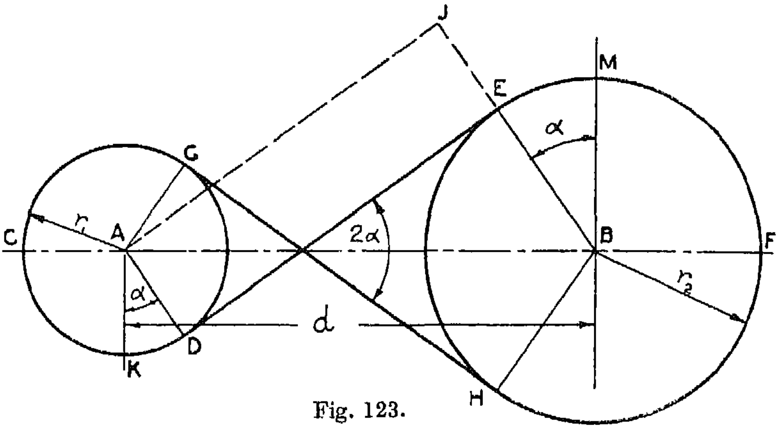


Fig. 123.

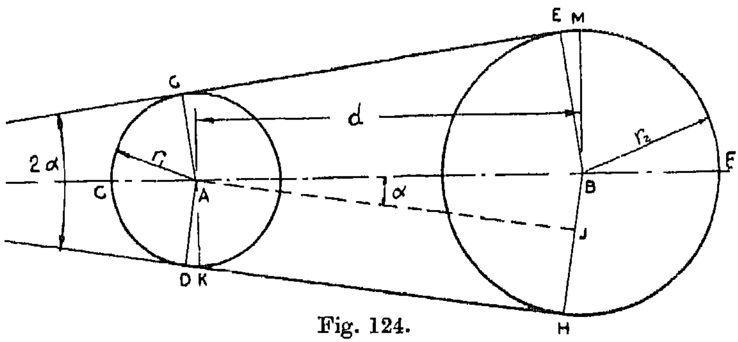


Fig. 124.

designing such pulleys is that the sum of the radii of corresponding steps shall be constant. This condition is, of course, in addition to that required for a given velocity ratio.

§ 150. **Length of Open Belt.** In Fig. 124, let 2α be the angle subtended by the sides of the belt. AJ is drawn parallel to DH, $\therefore BJ = BH - HJ = r_2 - r_1$.

Using the same notation as for a crossed belt,

$$\begin{aligned} l &= 2[\text{arc } CD + DH + \text{arc } HF] \\ &= 2[r_1(\tfrac{1}{2}\pi - \alpha) + AJ + r_2(\tfrac{1}{2}\pi + \alpha)] \\ &= 2[\tfrac{1}{2}\pi(r_1 + r_2) + \alpha(r_2 - r_1) + d\cos\alpha]. \end{aligned}$$

Now $\sin\alpha = \dfrac{BJ}{d} = \dfrac{r_2 - r_1}{d}$, and for pulleys of nearly equal size the angle α is small and $\sin\alpha$ may be written for α, also $\cos\alpha = \sqrt{(1 - \sin^2\alpha)}$.

Approximately, therefore,

$$\begin{aligned} l &= \pi(r_1 + r_2) + 2\frac{(r_2 - r_1)^2}{d} + 2d\sqrt{\left\{1 - \left(\frac{r_2 - r_1}{d}\right)^2\right\}} \\ &= \pi(r_1 + r_2) + 2\frac{(r_2 - r_1)^2}{d} + 2d\left(1 - \frac{1}{2}\frac{(r_2 - r_1)^2}{d^2}\right) \text{ approx.} \\ &= \pi(r_1 + r_2) + \frac{(r_2 - r_1)^2}{d} + 2d. \end{aligned}$$

In this case the length of belt depends upon $(r_1 + r_2)$ and upon $(r_2 - r_1)$, and for a belt connecting two stepped pulleys, the length of belt remains constant for equal tensions on different pairs of steps. If $(r_1 + r_2)$ is constant, then $(r_2 - r_1)$ is not constant and the simple relation holding for a crossed belt does not apply for an open belt.

EXAMPLE 2. Two parallel shafts, 14 ft. apart, are provided with pulleys 24 in. and 32 in. in diameter, and are connected by means of a crossed belt. It is desired to alter the direction of rotation of the driven shaft without altering that of the driving shaft. Find by how much the length of the belt should be altered. [*Inst. C. E.*]

For the crossed belt $\sin\alpha = \dfrac{r_1 + r_2}{d} = \dfrac{12 + 16}{14 \times 12} = \dfrac{1}{6}$,

$\therefore\ \alpha = 9°\ 36' = 0{\cdot}16755$ radian, $\cos\alpha = 0{\cdot}98600$.

$$\begin{aligned} l &= 2[(12 + 16)(\tfrac{1}{2}\pi + 0{\cdot}16755) + 14 \times 12 \times 0{\cdot}986] \\ &= 428{\cdot}64 \text{ in.} \end{aligned}$$

For the open belt,

$$\begin{aligned} l &= \pi(r_1 + r_2) + \frac{(r_2 - r_1)^2}{d} + 2d \\ &= \pi.28 + \frac{4^2}{14 \times 12} + 2 \times 14 \times 12 \\ &= 424{\cdot}06 \text{ in.} \end{aligned}$$

$\therefore$ Belt requires shortening $(428{\cdot}64 - 424{\cdot}06)$ or $4{\cdot}58$ in.

§ 151. Design of Stepped Pulleys. For a belt connecting two stepped pulleys, the length of belt is sensibly constant, and finding this length for two corresponding steps gives one relation between the radii. Another relation is that given by the required velocity ratio. For a crossed belt, the length of belt is constant when the sum of the radii of corresponding steps is constant.

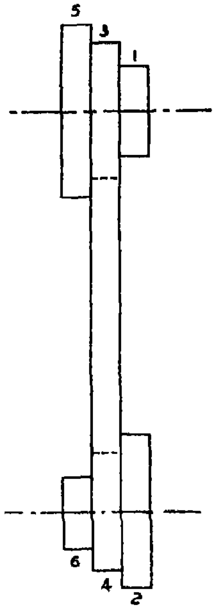


Fig. 125.

EXAMPLE 3. Design a set of stepped pulleys for driving a machine from a countershaft running at 160 revs. per min. The machine is to run at 60, 80, and 100 revs. per min., and the smallest step on the countershaft is 8 in. The distance between the centres of the two sets of pulleys is 6 ft. and the belt thickness may be neglected.

Take the two cases:

(*a*) for a crossed belt,
(*b*) for an open belt.

(*a*) Referring to Fig. 125, let 1, 2, 3, 4, 5, and 6 refer to the various steps. Step 1 on the countershaft corresponds to step 2 on the machine shaft.

$$\frac{N_2}{N_1} = \frac{r_1}{r_2}, \qquad r_1 = 4, \qquad \therefore\ r_2 = 4 \times \frac{160}{60} = 10{\cdot}67 \text{ in.},$$

$$r_1 + r_2 = 14{\cdot}67 \text{ in.}$$

$$\frac{N_4}{N_3} = \frac{r_3}{r_4} = \frac{80}{160} = \frac{1}{2},$$

also for a crossed belt, $r_3 + r_4 = 14{\cdot}67$, $\quad \therefore\ r_3 + 2r_3 = 14{\cdot}67$,

$$\therefore\ r_3 = 4{\cdot}89 \text{ in.},$$

$$r_4 = 9{\cdot}78 \text{ in.}$$

$$\frac{N_6}{N_5} = \frac{r_5}{r_6} = \frac{100}{160}, \quad \text{and} \quad r_5 + r_6 = 14{\cdot}67,$$

$$\therefore\ r_5 + 1{\cdot}6r_5 = 14{\cdot}67,$$

$$r_5 = 5{\cdot}64 \text{ in.}$$

$$r_6 = 9{\cdot}03 \text{ in.}$$

(*b*) For open belt, $r_1 = 4$ and $r_2 = 10{\cdot}67$ as before.

$$l = \pi(r_1+r_2)+\frac{(r_2-r_1)^2}{d}+2d$$

$$= \pi(14{\cdot}67)+\frac{(6{\cdot}67)^2}{72}+2\times72$$

$$= 190{\cdot}70 \text{ in.}$$

For middle steps, $r_4 = 2r_3$,

$$\therefore\ 190{\cdot}7 = \pi(r_3+r_4)+\frac{(r_4-r_3)^2}{d}+2d$$

$$= \pi(r_3+2r_3)+\frac{r_3^2}{72}+144,$$

$$\therefore\ \frac{r_3^2}{72}+9{\cdot}425r_3-46{\cdot}7 = 0,$$

$$r_3 = 4{\cdot}92 \text{ in.}$$

$$r_4 = 9{\cdot}84 \text{ in.}$$

For top speed $\qquad r_6 = 1{\cdot}6r_5$,

$$\therefore\ 190{\cdot}70 = \pi(r_5+1{\cdot}6r_5)+\frac{(0{\cdot}6r_5)^2}{72}+144,$$

$$\frac{r_5^2}{200}+8{\cdot}168r_5-46{\cdot}7 = 0,$$

$$r_5 = 5{\cdot}70 \text{ in.}$$

$$r_6 = 9{\cdot}12 \text{ in.}$$

§ **152. Shafts at Right Angles.** Two shafts at right angles to each other may be connected by belting, as in Fig. 126. The main point to be observed is that the approaching belt is in the plane in which the pulley is rotating. It will also be observed that the axis of one pulley is displaced by a distance equal to the radius of that pulley from the plane in which the other pulley is rotating.

§ **153. Ratio of Driving Tensions.** In Fig. 127 (*a*), let *BACD* represent a belt driving a pulley whose axis is *O*. The tight side tension T_1 is the part *AB* and the slack side tension is *CD*, and the angle of contact or angle of lap is $AOC = \theta$. The ratio of these tensions may be found by considering a small portion of the belt subtending an angle $\delta\theta$ at the centre of the pulley. Let *ab* represent this small portion of the belt and at *b* let T be the tension and δT the

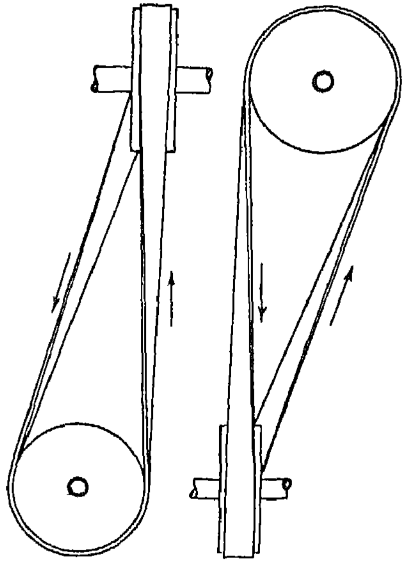
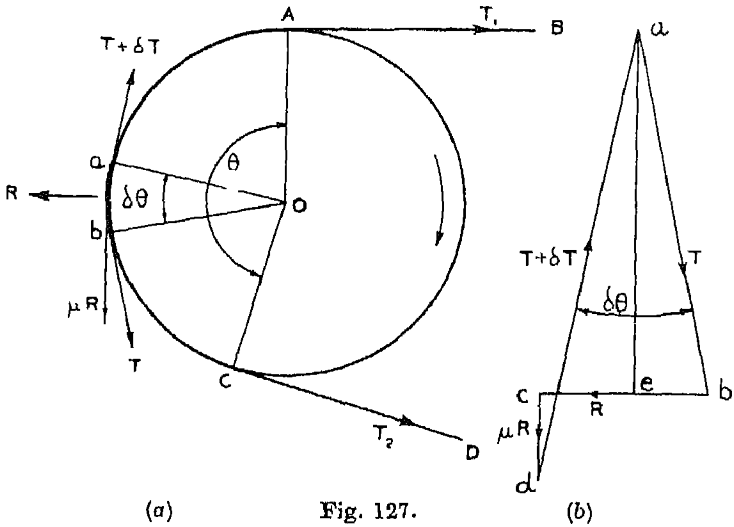


Fig. 126.



(a) Fig. 127. (b)

increase of tension over the arc ab, so that the tension at a is $T+\delta T$. Let R represent the normal reaction of the pulley on the small length of belt and μR, perpendicular to R, the

friction force. The maximum friction force will be when the belt is just on the point of slipping, for in this condition μ has its greatest value.

The small length of belt ab is in equilibrium (assuming uniform circular motion) under the action of the forces $T+\delta T$, T, R, and μR. These forces can be represented by vectors and for equilibrium the sum of the vectors is equal to zero. In Fig. 127 (*b*), ab is drawn parallel to T in Fig. 127 (*a*) and of length proportional to T. Similarly bc, cd, and da are drawn parallel to and proportional to R, μR, and $T+\delta T$ respectively. The angle bad is evidently equal to $\delta\theta$. When $\delta\theta$ is very small a perpendicular from a to bc, Fig. 127 (*b*), practically bisects the angle bad.

Resolving parallel to bc, all forces with components parallel to bc must balance one another, hence

$$R = (T+\delta T)\sin\tfrac{1}{2}\delta\theta + T\sin\tfrac{1}{2}\delta\theta.$$

Now $\delta\theta$ is a very small angle and $\sin\frac{1}{2}\delta\theta$ can be written equal to $\frac{1}{2}\delta\theta$. Further, the product of δT and $\frac{1}{2}\delta\theta$ is a quantity of the second order of small magnitudes and may be neglected.

$$\therefore\ R = \tfrac{1}{2}T\,\delta\theta + \tfrac{1}{2}T\,\delta\theta = T\,\delta\theta.$$

Resolving perpendicular to bc,

$$\mu R = (T+\delta T)\cos\tfrac{1}{2}\delta\theta - T\cos\tfrac{1}{2}\delta\theta.$$

Now $\cos\frac{1}{2}\delta\theta$ is practically equal to unity, and

$$\mu R = \delta T.$$

Substituting the value of R in this equation,

$$\mu . T\,\delta\theta = \delta T,$$

or

$$\mu\,\delta\theta = \frac{\delta T}{T}.$$

Integrating this expression between the appropriate limits,

$$\int_0^\theta \mu\,d\theta = \int_{T_2}^{T_1} \frac{dT}{T},$$

$$\mu\theta = [\log_e T]_{T_2}^{T_1} = \log_e T_1 - \log_e T_2 = \log_e \frac{T_1}{T_2}. \qquad (1)$$

Taking the antilogarithm of each side and transposing,

$$\frac{T_1}{T_2} = e^{\mu\theta}.$$

e is the base of the Napierian or hyperbolic logarithms.

θ is the angle of contact in radians, and for a belt connecting two pulleys the least value of θ will give the least value of $\frac{T_1}{T_2}$. The angle of contact is readily found when the inclination of the belt to the line joining the centres of the pulleys is known, and the method of finding this for open and crossed belts has been described in §§ 149 and 150.

Equation (1) above involves finding the antilogarithm to base e, and since logarithms to base 10 are much more convenient, it is an advantage to express this equation in logarithm form to base 10.

$$\log_e \frac{T_1}{T_2} = \mu\theta.$$

Hyperbolic logarithms are converted to common logarithms by dividing the hyperbolic logarithm by 2·3026, which is equivalent to multiplying by 0·4343, thus:

$$\frac{1}{2{\cdot}3026} \log_e \frac{T_1}{T_2} = \log_{10} \frac{T_1}{T_2} \quad \left(\text{usually written } \log \frac{T_1}{T_2}\right),$$

$$\log \frac{T_1}{T_2} = \frac{\log_e \frac{T_1}{T_2}}{2{\cdot}3026} = \frac{\mu\theta}{2{\cdot}3026} = 0{\cdot}4343\mu\theta,$$

$$\text{i.e. } \log \frac{T_1}{T_2} = 0{\cdot}4343\mu\theta.$$

§ 154. **Power Transmitted by Belts.** The effective tension or force acting at the circumference of a pulley is $(T_1 - T_2)$, and the work done per minute is the product of $(T_1 - T_2)$ and the speed of the belt.

Let V = speed of belt in feet per minute;

$T_1 - T_2$ = effective tension in lb.;

$$\text{H.P. transmitted} = \frac{(T_1 - T_2)V}{33{,}000}.$$

§ 155. **Width of Belt.** To find the width of belt to transmit a given horse-power, the ratio $\frac{T_1}{T_2}$ can be calculated if the

angle of lap and the coefficient of friction between the belt and pulley are known. The effective tension (T_1-T_2) can be calculated if the speed and horse-power are known, and from these two relations values of T_1 and T_2 can be found. T_1 is the greatest pull in the belt and approximately determines the width of belt required.

Let T_1 = tight side tension in lb.,

t = safe tension per inch of width,

b = breadth or width of belt.

Then $$b = \frac{T_1}{t}.$$

The maximum tension in a belt is only equal to T_1 at moderate speeds. At high speeds the centrifugal tension becomes of consequence and should be included when determining the width of belt. The above expression for width of belt should, therefore, only be used for slow or moderate speeds.

Example 4. The driving pulley on a main shaft is 24 in. in diameter and rotates at 120 revs. per min. A countershaft is to be driven at 300 revs. per min. by means of an open belt. The distance apart of the shaft centres is 8 ft. and the coefficient of friction is 0·3. Find the width of belt required to transmit 4 horse-power if the tension is not to exceed 80 lb. per in. of width.

$$\frac{N_2}{N_1} = \frac{r_1}{r_2}, \qquad \therefore\ r_2 = 12\times\frac{120}{300} = 4{\cdot}8 \text{ in.}$$

Referring to Fig. 124, $\sin\alpha = \dfrac{12-4{\cdot}8}{8\times 12} = \dfrac{7{\cdot}2}{8\times 12} = 0{\cdot}075,$

$$\therefore\ \alpha = 4^\circ\ 18' = 0{\cdot}07505 \text{ radian.}$$

Least value of $\theta = \pi-2\alpha = 3{\cdot}1416-0{\cdot}1501 = 2{\cdot}9915.$

$$\log\frac{T_1}{T_2} = 0{\cdot}4343\times 0{\cdot}3\times 2{\cdot}9915 = 0{\cdot}38977,$$

$$\therefore\ \frac{T_1}{T_2} = 2{\cdot}453,$$

or $$T_1 = 2{\cdot}453T_2.$$

$$\text{H.P.} = \frac{(T_1-T_2)V}{33{,}000} = \frac{(2{\cdot}453T_2-T_2)\pi\,.\,2\,.\,120}{33{,}000} = 4.$$

$$\therefore\ T_2 = 120{\cdot}6, \qquad T_1 = 296 \text{ lb.}$$

Hence width of belt $= \dfrac{296}{80} = 3{\cdot}7$ in.

EXAMPLE 5. Find the number of turns a rope must be coiled round a vertical post to hold a pull of 5,000 lb. if the pull on the free end is 10 lb. Coefficient of friction 0·2.

$$\log \frac{T_1}{T_2} = 0{\cdot}4343\mu\theta.$$

$$\log \frac{5{,}000}{10} = 0{\cdot}4343 \times 0{\cdot}2 \,.\, \theta$$

$$\theta = 31{\cdot}1 \text{ radians.}$$

Number of turns $= \dfrac{31{\cdot}1}{2\pi} = 4{\cdot}95.$

§ **156. Centrifugal Tension.** When a particle is rotating in a circular path it has been seen that the centripetal acceleration is $\omega^2 r$ or $\dfrac{v^2}{r}$, where r is the radius of the path. In a belt moving in a circular path with a pulley, considering a small portion of the belt ab, Fig. 128 (a), this small portion has a centripetal acceleration of $\dfrac{v^2}{r}$ and the force causing this is the centripetal force. This centripetal force is applied by the radial inward components of a tension T_3 acting at a and b. The centrifugal force acting on ab acts radially outwards and is represented by R. The components of T_3 acting radially inwards balance the centrifugal force R acting outwards. The tension T_3 is termed *centrifugal tension* and is due to the speed of the belt moving in a circular path and is independent of T_1 and T_2. Since the centrifugal tension depends only on the speed of the belt in a circular path, T_3 may be found by considering the forces acting on the length ab when no power is being transmitted and hence the forces represented in the figure are only applicable to these conditions, viz. that the belt is not transmitting any power.

In Fig. 128 (a), consider a small length of belt subtending an angle $\delta\theta$ at the centre of the pulley.

Let w = weight per foot length of belt,

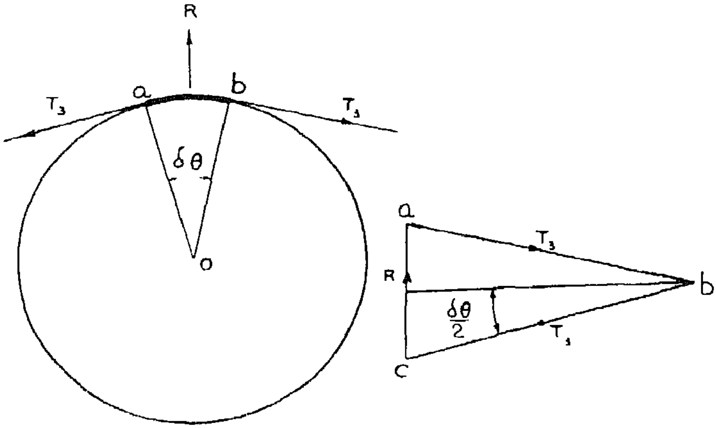
r = radius of pulley in feet,

v = linear speed of belt in feet per second.

Weight of small element of belt $= wr\,\delta\theta$.

Centrifugal force, $R = \dfrac{w.r.\delta\theta}{g} \cdot \dfrac{v^2}{r}$.

Let T_3 = centrifugal tension, which is constant throughout for that part of the belt in contact with the pulley.



(a) Fig. 128. (b)

The element of the belt is in equilibrium due to the forces T_3 acting tangentially at a and at b and to the force R. The triangle of forces is drawn at (b) in which ab and bc are drawn parallel to and proportional to T_3, and ca is drawn parallel to R.

Resolving parallel to ac,

$$R = T_3 \sin\frac{\delta\theta}{2} + T_3 \sin\frac{\delta\theta}{2} = T_3\,\delta\theta,$$

if $\delta\theta$ is a very small angle:

$$\therefore \quad \frac{wr\,\delta\theta}{g}\,\frac{v^2}{r} = T_3\,\delta\theta,$$

or
$$T_3 = \frac{wv^2}{g}.$$

If the tensions T_1 and T_2 are calculated for the transmission of a given horse-power at a given speed, the centrifugal tension is additional to these tensions and the actual tensions

become T_1+T_3 and T_2+T_3. When the effect of centrifugal tension is taken into account, the greatest tension in the belt is T_1+T_3 and the width of belt should be modified to include this.

EXAMPLE 6. In a belt drive the ratio of the tensions T_1 and T_2 may be taken as 2·2 when the effect of centrifugal tension is neglected. The speed of the belt is 3,000 ft. per min. and the safe tension is not to exceed 200 lb. per sq. in. Find the width of belt to transmit 10 horse-power if the belt weighs 0·04 lb. per cub. in. and its thickness is $\frac{3}{8}$ in.

$$\text{H.P.} = \frac{(T_1-T_2)V}{33{,}000},$$

$$\therefore\ T_1-T_2 = \frac{10\times 33{,}000}{3{,}000} = 110 \text{ lb.}$$

$$T_1 = 2{\cdot}2T_2,$$

$$\therefore\ 2{\cdot}2T_2-T_2 = 110, \qquad T_2 = 91{\cdot}7,$$

$$T_1 = 202 \text{ lb.}$$

Let b = width of belt in inches,

$$\text{weight per foot length} = b\times 12\times \tfrac{3}{8}\times 0{\cdot}04 = 0{\cdot}18b,$$

$$T_3 = \frac{wv^2}{g} = \frac{0{\cdot}18b(50)^2}{32{\cdot}2} = 14b.$$

$$\text{Maximum tension} = T_1+T_3 = 202+14b,$$

$$\therefore\ b\times \tfrac{3}{8}\times 200 = 202+14b,$$

$$\therefore\ b = 3{\cdot}31 \text{ in.}$$

§ **157. Maximum Power for a Belt.** The greatest tension T to which a given belt may be subjected is a fixed amount, and when the belt is used to its maximum capacity this tension will be equal to T_1+T_3. The centrifugal tension T_3 increases with the square of the linear speed and the available value of T_1 thus decreases, since $T_1 = T-T_3$. The horse-power transmitted increases with increase of linear speed, but the available value of T_1 and consequently of (T_1-T_2) diminishes. Hence, the horse-power transmitted increases with the increase of speed up to a certain point, after which it decreases. There is thus a certain velocity for a belt at which maximum horse-power may be transmitted.

Let T = maximum permissible tension in a given belt,
T_1 = tension on tight side,
T_2 = tension on slack side,
T_3 = centrifugal tension,
v = linear speed in feet per second.

$$\text{H.P.} = \frac{(T_1-T_2)v}{550} \quad \text{and} \quad \frac{T_1}{T_2} = e^{\mu\theta},$$

$$\therefore \text{ H.P.} = \frac{\left(T_1-\dfrac{T_1}{e^{\mu\theta}}\right)v}{550} = c.T_1 v,$$

where
$$\frac{1-\dfrac{1}{e^{\mu\theta}}}{550} = c = \text{constant}.$$

Now, $T = T_1+T_3$ when the belt is transmitting maximum horse-power.

$$\therefore \text{ H.P.} = c.v(T-T_3) = cv\left(T-\frac{wv^2}{g}\right).$$

Differentiating this expression with respect to v,

$$\frac{d(\text{H.P.})}{dv} = c\left(T-\frac{3wv^2}{g}\right) = 0 \text{ for a maximum},$$

$$\therefore \ T = \frac{3wv^2}{g} = 3T_3,$$

and
$$T_1 = T-T_3 = 3T_3-T_3 = 2T_3.$$

The velocity at which maximum horse-power is transmitted is given by $v = \sqrt{\dfrac{gT}{3w}}$. Maximum horse-power is transmitted when one-third of the maximum tension allowed in a belt is utilized as centrifugal tension, or conversely when the tight side tension is equal to twice the centrifugal tension.

EXAMPLE 7. Find the speed at which a belt 3 in. wide weighing 1·2 lb. per sq. ft. will transmit maximum horse-power, if the safe permissible tension is 70 lb. per in. of width. If the ratio of T_1 to T_2 is 2·1, find the magnitude of the maximum horse-power that can be transmitted by this belt.

Weight per foot length $= 1{\cdot}2\times\frac{3}{12} = 0{\cdot}3$ lb.

$$T_3 = \frac{wv^2}{g} = \tfrac{1}{3}T,$$

$$\therefore \ 0{\cdot}3\times\frac{v^2}{32{\cdot}2} = \frac{70}{3}\times 3.$$

$$v^2 = 7{,}520,$$

$$\therefore\ v = 86{\cdot}7 \text{ ft. per sec.}$$

$$T_1 = \tfrac{2}{3}T = \tfrac{2}{3} \times 210 = 140,$$

$$\therefore\ T_2 = \frac{140}{2{\cdot}1} = 66{\cdot}7.$$

$$T_1 - T_2 = 73{\cdot}3,$$

$$\therefore\ \text{maximum H.P.} = \frac{73{\cdot}3 \times 86{\cdot}7}{550} = 11{\cdot}55.$$

§ **158. Rope Drives.** Ropes of cotton, hemp, or manilla may be used in place of belts where a constant drive is required, as from an engine to a main shaft. The ropes run in grooves formed on the rims of the pulleys, and for this reason are not suitable for use on stepped pulleys. Cotton ropes are more generally used on account of their better characteristics. Ropes are very suitable for the transmission of large powers, especially when several main shafts on different floors of a factory are driven from the same engine. For cases of this nature as many as 50 or 60 ropes may be used, and the pulley on the crank-shaft contains as many grooves as there are ropes. For cotton, hemp, or manilla ropes the groove is tapered, as shown in Fig. 129 (*a*), and the rope is gripped by the sides of the groove. The angle of the groove is about 45°, and owing to the wedging action it is unnecessary to give the ropes a large initial tension as in the case of belts. Wire ropes, built up of twisted strands of wire, may be used at comparatively low speeds where great tension is required. A wire rope should run on the bottom of the groove, as the gripping action of the sides would soon cause serious wear.

In Fig. 129 (*a*), let 2α be the angle of the groove on a rope pulley and R_1 the total normal reaction of the sides of the groove on the rope. If the groove is symmetrical, this is equivalent to $\frac{1}{2}R_1$ at each inclined surface. The reaction of the pulley in the plane of the groove is found by taking the components of the two normal reactions $\frac{1}{2}R_1$ in the plane of the groove. From Fig. 129 (*b*) it is seen that R, the reaction in the plane of the groove, is equal to the components of $\frac{1}{2}R_1$ in that direction:

$$\tfrac{1}{2}R_1 \times \sin\alpha = \tfrac{1}{2}R,$$

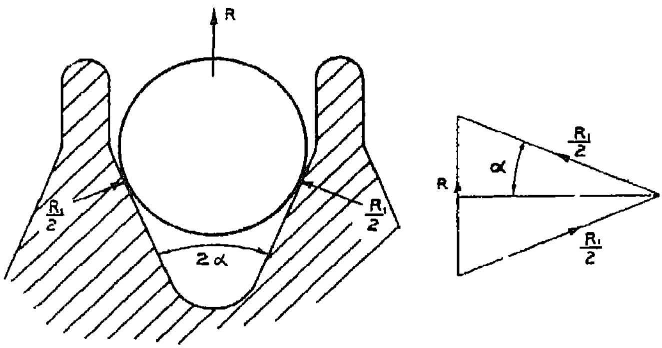
$$\therefore \quad R = R_1 \sin\alpha.$$

The friction force is μR_1 or $\dfrac{\mu R}{\sin\alpha}$ and the investigation for finding the ratio of T_1 to T_2 is on similar lines to that for a belt. Referring to Fig. 127, and using the same notation,

$$R = T\,\delta\theta,$$

and

$$\mu R_1 = \delta T.$$



(a) Fig. 129. (b)

Substituting for R_1, $\dfrac{\mu R}{\sin\alpha} = \delta T$,

$$\therefore \quad \frac{\mu . T\,\delta\theta}{\sin\alpha} = \delta T$$

or

$$\frac{\delta T}{T} = \frac{\mu . \delta\theta}{\sin\alpha}.$$

Integrating this expression as before,

$$\frac{T_1}{T_2} = e^{\frac{\mu\theta}{\sin\alpha}}$$

or

$$\log\frac{T_1}{T_2} = 0{\cdot}4343\,\frac{\mu\theta}{\sin\alpha}.$$

This expression gives the value of $\dfrac{T_1}{T_2}$ for each rope. The horse-power transmitted by a number of ropes is

$$\text{H.P.} = \frac{n.(T_1 - T_2)V}{33{,}000},$$

where n is the number of ropes.

EXAMPLE 8. A rope pulley having a mean diameter of 5 ft. centre to centre of the ropes rotates at 90 revs. per min. The angle of lap of the ropes is 190° and the angle of the groove is 45°. The safe tension per rope is 150 lb. and the coefficient of friction between the rope and the sides of the groove is 0·25. Find the number of ropes required to transmit 170 horse-power.

$$\log\frac{T_1}{T_2} = 0{\cdot}4343\frac{\mu\theta}{\sin\alpha} = 0{\cdot}4343 \times 0{\cdot}25 \times \frac{190}{180} \times \frac{\pi}{0{\cdot}38268}.$$

$$\frac{T_1}{T_2} = 8{\cdot}726.$$

$$T_1 = 150, \quad \therefore\ T_2 = \frac{150}{8{\cdot}726}.$$

$$T_1 - T_2 = 150 - \frac{150}{8{\cdot}726} = 132{\cdot}8.$$

$$\text{H.P.} = \frac{n.(T_1 - T_2)V}{33{,}000},$$

$$\therefore\ n = \frac{170 \times 33{,}000}{132{\cdot}8 \times \pi \times 5 \times 90} = 29{\cdot}9, \text{ say } 30.$$

EXERCISES. IX

1. A horizontal shaft running at 200 revs. per min. is to drive a parallel shaft at 300 revs. per min. The pulley on the driving shaft is 18 in. diameter. If the belt is $\frac{1}{4}$ in. thick, find the diameter of the driven pulley: (*a*) neglecting belt thickness; (*b*) taking belt thickness into account; (*c*) assuming in the latter case a slip of 4 per cent.

2. A shaft running at 100 revs. per min. is to drive another shaft at 280 revs. per min. The distance between the shaft centres is 10 ft. Find the length of open belt required, if the smaller pulley is 12 in. diameter.

3. Find the horse-power transmitted by the belt in Question 2 if the width of the belt is 6 in. and the allowable stress per inch width is 70 lb. Coefficient of friction = 0·3.

4. A belt 6 in. by $\frac{1}{4}$ in. connects two pulleys 4 ft. in diameter running at 300 revs. per min. Find the centrifugal tension per square inch and per inch width. The belt weighs 72 lb. per cub. ft.

5. Two parallel shafts, 10 ft. apart, are to have a velocity ratio of 6, and are to be connected by belting. The pulley on the quick-running shaft is 8 in. in diameter.

Determine the length of belting required and also the pulley diameter necessary for giving a velocity ratio of unity when using the same length of belting: (*a*) if the belt is open; (*b*) if the belt is crossed. [*I. Mech. E.*]

6. Leather belting 3 in. wide is used to transmit power between two parallel shafts which run at the same speed. The pulleys are 24 in. in diameter. At slow speeds the tension in the slack side is half the tension in the driving side, and the angle of lap on the pulley is 180°. The maximum load on the belting must not exceed 80 lb. per in. of width. Neglecting the effect of centrifugal force, estimate the maximum horse-power which can be transmitted at 550 revs. per min.

Show that, owing to centrifugal action, the actual power transmitted will be less than this estimate, and calculate this actual power if the weight of the belting is 0·25 lb. per ft. and the angle of lap at this speed is observed to be 140°. [*I. Mech. E.*]

7. A belt running on a pulley with a Vee groove is required to transmit power. The groove contains an angle of 60°, and the radius of the centre line of the belt as it passes over the pulley is 3 in., and the angle of lap is 120°.

Determine the least possible driving tensions in the belt when transmitting 4 horse-power at 1,600 revs. per min., assuming a coefficient of friction of 0·25. [*I. Mech. E.*]

8. A shaft running at 200 revs. per min. is required to drive another shaft parallel to it and distant 9 ft. from it. The drive is to be made by an open belt, and the speed cones so proportioned that the driven shaft can be run at 100, 140, or 180 revs. per min. If the minimum speed cone diameter is 9 in., determine the other speed cone diameters. [*I. Mech. E.*]

9. Design a set of speed pulleys for driving a lathe at 50, 70, and 90 revs. per min. from a countershaft running at 110 revs. per min. The belt is to be 3 in. wide and crossed, and the distance between the axes of countershaft and lathe spindle is 9 ft. The smallest pulley diameter is to be $4\frac{1}{2}$ in.

Make a sketch showing in longitudinal section the stepped cone pulley for the lathe spindle. [*I. Mech. E.*]

10. Two horizontal shafts, 40 ft. apart, are required to run at 250 and 200 revs. per min. and are to be connected by a rope drive. The rope is to be $1\frac{1}{2}$ in. in diameter, the grooves on the pulleys are to have an angle of 60°, the angle of lap may be taken as 180°, and the coefficient of friction between rope and pulley as 0·2.

If the linear speed of the rope is 80 ft. per sec., and the tension on a rope is limited to 400 lb., how many ropes will be required to transmit 150 horse-power?

Sketch the pulley for the slower running shaft. [*I. Mech. E.*]

11. A band brake is to be provided for a winch. The diameter of the barrel of the winch is 18 in. and the maximum loading for the winch is 2 tons. The angle of lap for the brake strap is 270°, the brake wheel diameter 24 in., and the coefficient of friction can be taken as 0·15. Assuming a working stress of 5 tons per sq. in., determine the

width of brake strap $\frac{3}{16}$ in. thick necessary to hold the load, and sketch the arrangement of brake and brake handle. [*I. Mech. E.*]

12. Power has to be transmitted by means of leather belting between two horizontal shafts which are at right angles to each other and separated by a vertical distance of 15 ft. The upper shaft, which is the driving shaft, makes 180 revs. per min., and the driven shaft is required to make 300 revs. per min.

Obtain suitable diameters for the pulleys, and also the width of belting for the transmission of 5 horse-power. The load in the belting must not exceed 80 lb. per in. of width.

Clearly show by sketches how the pulleys should be placed. [*I. Mech. E.*]

13. Stepped pulleys are required for a leather belting drive connecting two parallel shafts 12 ft. apart, and running in opposite directions. The driving shaft has a speed of 150 revs. per min., and the driven shaft is required to run at 140, 200, and 260 revs. per min. The maximum horse-power to be transmitted is 4, and the load in the belting must not exceed 80 lb. per in. of width. Design suitable stepped pulleys for the two shafts, and specify the length and width of belting required. [*I. Mech. E.*]

14. Power is transmitted from a small turbine pulley to a countershaft by means of an open leather belt. The pulley on the turbine shaft is $6\frac{1}{4}$ in. in diameter, and the pulley on the countershaft is 42 in. in diameter, and the axes of the shafts are parallel and 6 ft. apart. If the coefficient of friction between belt and pulley is 0·2, what is the maximum torque which can be given to the countershaft when the turbine is starting up, so that the tension in the belt shall not exceed 150 lb.?

When the turbine pulley is running at 2,800 revs. per min. it is found that the belt begins to slip when transmitting 4·7 horse-power. Find the corresponding torque on the countershaft, and explain why it is less than the starting torque. [*I. Mech. E.*]

15. Find the number of times a hauling-rope must be wound round a rotating capstan in order to haul 10 trucks, each weighing 30 tons. Rolling resistance, 12 lb. per ton. Pull on free end of rope, 20 lb.; coefficient of friction 0·4. [*Inst. C. E.*]

16. The tup of a drop-forging plant is sometimes lifted by means of a belt passed over a revolving pulley on a main shaft running above the hammers. The belt over the pulley is arranged so that one end is attached to the tup, whilst the other end hangs free and within reach of the smith operating the hammer.

Suppose the arc of contact between the belt and the pulley is 180°, and that the coefficient of friction between the leather and the pulley is 0·5, find what weight of tup could be lifted when the smith exerts a pull of 50 lb. on the free end of the belt. [*Inst. C. E.*]

17. Prove that the ratio between the tensions on the tight and slack sides of a belt transmitting power between two pulleys is $e^{\mu\theta}$, where

μ = coefficient of friction between belt and pulley, and
θ = minimum arc of contact between belt and pulley.

Explain clearly why the belt is re-tightened when it has stretched unduly after service. [*Inst. C. E.*]

18. Calculate the width of belt required to transmit 5 horse-power under the following conditions:

Revolutions per minute of pulley . . .	= 300
Diameter of pulley	= 18 in.
Angle of contact of shaft	= 160°
Coefficient of friction	= 0·3
Maximum tension per inch width of belt . .	= 70 lb.

19. Determine the length of a crossed belt to connect the pulleys on two shafts which are 18 ft. apart, when the diameters of the pulleys are $4\frac{1}{2}$ ft. and $2\frac{1}{2}$ ft. respectively. Prove any formula you use.

Show by sketches how motion may be transmitted by a belt passing over two pulleys only on two shafts which are at right angles but in different planes. Distinguish between the driving and the driven shafts. [*Inst. C. E.*]

20. Find the horse-power transmitted by a belt when running under the following conditions:

Diameter of pulley	= 2 ft.
Revolutions per minute . . .	= 150
Angle embraced by belt . . .	= 150°
Coefficient of friction . . .	= 0·35
Maximum pull on belt . . .	= 700 lb.

21. Determine the width of a belt $\frac{1}{4}$ in. thick for the transmission of 20 horse-power, when the following assumptions are made:

Peripheral velocity of belt . . .	= 3,000 ft. per min.
Angle of contact of belt on pulley . .	= 180°
Coefficient of friction between belt and pulley	= 0·5
Minimum value of $f_1+\frac{1}{2}f_2$ . . .	= 240 lb. per sq. in.

where f_1 = tensile stress on tight side of belt, and
f_2 = tensile stress on slack side of belt.

Assuming that the initial tension in a belt is equal to the mean of the tensions on the tight and slack sides, determine also the minimum and maximum values of the initial tension to which this belt should be subjected if the maximum value of $f_1+\frac{1}{2}f_2$ is not to exceed 320 lb. per sq. in. [*Inst. C. E.*]

22. Find the number of times a hauling rope must be wound round a rotating capstan in order to haul 10 trucks, each weighing 30 tons, up a gradient of 1 in 30. Rolling resistance, 10 lb. per ton. Pull on free end of rope, 40 lb.; coefficient of friction between rope and drum = 0·4. [*Lond. B.Sc.*]

23. Sketch and describe an arrangement of pulleys and belts for driving a shaft from another shaft parallel to it so as to give a quick return motion.

A shaft running at 150 revs. per min. drives a parallel shaft at 300 revs. per min., the pulley on the first shaft being 3 ft. in diameter. It is required to drive the second shaft at 100 revs. per min., using the same belt. What should be the diameters of the pulleys if the belt is open and the distance between the shafts is 8 ft.? [*Lond. B.Sc.*]

24. A shaft running at 100 revs. per min. has to drive by a belt another shaft at 250 revs. per min., the pulley on the first shaft being 30 in. in diameter. The shafts are 8 ft. apart, and the belt is crossed. If the belt has to transmit 16 horse-power, and if the coefficient of friction is 0·20, find the necessary width of belt if the maximum permissible tension is 80 lb. per in. of width. [*Lond. B.Sc.*]

25. A motor drives a pump by means of a belt on a 5-ft. diameter pulley. The pump runs at 150 revs. per min. and delivers 600 gal. per min. against a head of 125 ft. The contact between the belt and the pulley is 180°, and the coefficient of friction 0·25. The pull on the belt is not to exceed 80 lb. per in. of width. Assuming that the overall efficiency of the pump is 60 per cent., determine the dimensions of the belt and the horse-power of the motor. [*Lond. B.Sc.*]

26. Calculate the centrifugal tension in a belt which runs over two pulleys at a speed of 5,500 ft. per min. The belt is 8 in. wide, $\frac{5}{16}$ in. thick, and weighs 0·036 lb. per cub. in.

If $e^{\mu\theta} = 2$, and the maximum permissible tension on the belt is 250 lb. per sq. in., find the horse-power that can be transmitted at the above speed. [*Lond. B.Sc.*]

27. A belt 4 inches wide running at 2,500 feet per minute connects two pulleys, the minimum angle of lap being 170 degrees. If the coefficient of friction between the belt and the surfaces of the pulleys is 0·2 find the greatest horse-power that can be transmitted when the safe tension in the belt is 80 lb. per inch of width.

If, by the application of belt dressing, the coefficient of friction is increased to 0·25 find the percentage increase in speed to transmit 20 per cent. more power when the safe stress is reduced to 70 lb. per inch of width. [*Lond. B.Sc.*]

28. The driving pulley on a main shaft running at 250 r.p.m. is 3 feet in diameter and drives by means of a belt a countershaft whose speed is 700 r.p.m. The minimum angle of lap may be taken as 170°, the coefficient of friction between the belt and the pulley surfaces as 0·25, the weight per cubic inch of belt 0·04 lb., the safe tension in the belt, including centrifugal tension, 300 lb. per square inch of cross section.

Find the width of belt $\frac{1}{4}$ inch thick to transmit 8 horse-power. [*Lond. B.Sc.*]

29. In a rope-drive transmitting 1,200 horse-power, the flywheel has an effective diameter of 12 feet and a speed of 75 revolutions per minute. There are 32 ropes running in grooves which have an angle of 45 degrees. If the limiting coefficient of friction between the ropes and the sides of the grooves is 0·25, the angle of lap of the ropes is 180 degrees and the weight per foot length of rope is 1·08 lb., find the greatest tension in each rope.

Find the increase in speed to transmit 20 per cent. more power if the greatest tension is to remain unaltered. [*Lond. B.Sc.*]

CHAPTER X

FLY-WHEELS AND TURNING-MOMENT DIAGRAMS

§ **159. Function of Fly-wheel.** In an ordinary direct-acting steam engine, the steam is usually admitted for a portion of each stroke, and the pressure of the steam on the piston is not constant throughout the stroke. Neglecting for the moment inertia effects, and considering a horizontal engine as in Fig. 116, the force available at the crosshead is equal to the pressure on the piston. In § 139 it was shown that the turning-moment on the crank-shaft is equal to $P \times CD$, CD being the intercept, on the vertical through the crank-shaft centre, cut off by the connecting-rod produced and the crank-shaft centre C. It will be readily seen that for different crank angles the intercept CD will vary in magnitude from zero, when the crank is on the inner dead centre, to a maximum when the crank and connecting-rod are approximately at right angles to each other. Since both P and CD vary, the turning-moment varies considerably during each stroke.

This variation of turning-moment means a variation of the rate at which work is done by the engine, with a consequent variation of speed throughout the stroke. In many cases it is essential that the speed of the engine is constant within small limits, and in these circumstances it is usual to fit a fly-wheel to keep the speed within the prescribed limits. The rate at which work is taken from the engine is usually constant for such short intervals of time as that required for a stroke, and at certain periods of the stroke the engine is developing more power than is being taken from it, while at other periods the engine is developing less power. The fly-wheel acts as a reservoir which can give up and absorb energy as required When a fly-wheel gives up energy its speed decreases, and conversely, when energy is absorbed the speed increases. This alternate taking in and giving out of energy is continuous, and occurs at least once during each stroke. The fly-wheel

thus tends to keep the speed of the engine within the required limits from revolution to revolution.

If the load on the engine alters, the speed of the engine will decrease or increase according to whether the load is greater or less than the mean load. This means a definite alteration of speed unless the quantity of steam or the pressure is altered in such a manner as to equalize the work done by the engine and the work taken from the engine. This equalization is made possible by the use of a governor. The governor and the fly-wheel are each used for the purpose of keeping the speed within certain limits, but the function of each is quite different. The action of the fly-wheel is continuous and regulates the speed over exceedingly short intervals of time, while the governor is called into action for variation of load or of steam pressure, and is, to some extent, intermittent.

§ **160. The Indicator Diagram.** An indicator diagram is an autographic representation of the variation of the fluid pressure in an engine cylinder throughout the stroke. In Fig. 130 (*a*) the diagrams illustrate typical indicator diagrams for a horizontal double-acting steam engine. In Fig. 130 (*b*) a piston is shown in its cylinder. For convenience of definition, the outstroke is regarded as that in which the piston rod moves out of the cylinder, and the instroke as that in which the piston rod moves into the cylinder. When the piston is moving to the left it is moving on the outstroke, and the cycle of operations which takes place on the side of the piston marked A is known as the outstroke cycle; when the piston is moving to the right the piston is moving on the instroke, and the cycle of operations which takes place on the side B is the instroke cycle. The indicator diagram for the outstroke cycle is $abcd$ in Fig. 130 (*a*) and for the instroke cycle is $efgh$.

Consider the piston in the position shown in Fig. 130 (*b*), and assume that it is moving to the left. The pressure per square inch on the side A is represented by mb, and the pressure on side B by mh. The net pressure on the piston for this position when moving to the left is $mb - mh = hb$. When the piston is in the same position and moving to the right, the

pressure on the side B is mf and on the side A, md. The net pressure for this position is $mf - md = df$.

§ 161. Diagram of Effective Steam Pressure. The ordinates hb and df, Fig. 130 (a), represent the net or effective

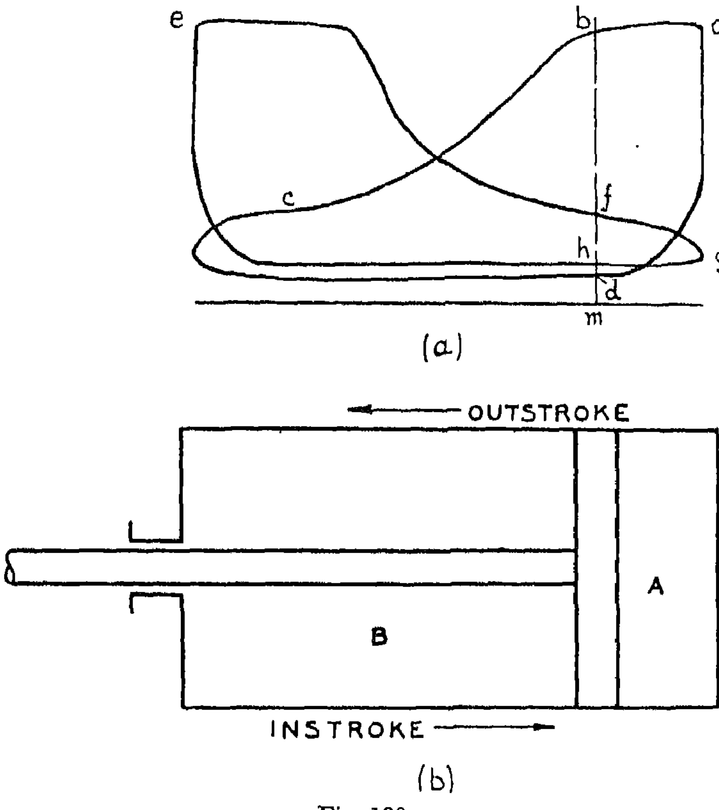


Fig. 130.

pressure in pounds per square inch on the two sides of the piston for the given position. If these ordinates are plotted on a new base line MN, Fig. 131 (a) and (b), for the corresponding piston position, points are obtained which lie on a diagram of effective pressure. By plotting similar ordinates for other piston positions the complete diagram is obtained. In

Fig. 131 (*a*) the diagram for the outstroke is $NabcMN$ and in Fig. 131 (*b*) the diagram for the instroke is $MafbNM$.

§ **162. Correction of Vertical Engine.** In a vertical engine the weight of the reciprocating parts assists the steam pressure on the down or outstroke and opposes the steam pressure on the up or instroke.

Let W = total weight of reciprocating parts in pounds,

A = area of piston in square inches;

then $\frac{W}{A}$ = weight of reciprocating parts per square inch of piston area.

The correction for the weight of the reciprocating parts is made by increasing the effective steam pressure on the piston by $\frac{W}{A}$ for the outstroke and decreasing it by the same amount for the instroke. In Fig. 131 (*a*) and (*b*) a new base line $M_1 N_1$ is drawn parallel to MN, and $MM_1 = NN_1 = \frac{W}{A}$. The effective pressure is now measured from the new base line $M_1 N_1$. The net pressures for the given position of the piston are represented by bd and fe respectively. For a vertical engine the base line is $M_1 N_1$ and for a horizontal engine it is MN.

§ **163. Correction for Accelerating Forces.** The acceleration of the reciprocating parts may be found either by calculation or by one of the geometrical constructions described in Chapter V. For any crank angle θ from the inner dead centre, the acceleration is given by

$$\omega^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right).$$

The accelerating force required to produce this acceleration is $\frac{W}{g}\omega^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right)$ pounds, and the accelerating force per square inch of piston area is $\frac{W}{gA}\omega^2 r\left(\cos\theta + \frac{r}{l}\cos 2\theta\right)$. Since the acceleration is positive for the earlier part of the stroke, the accelerating force opposes the piston effort, and for the latter part assists the piston effort when the acceleration

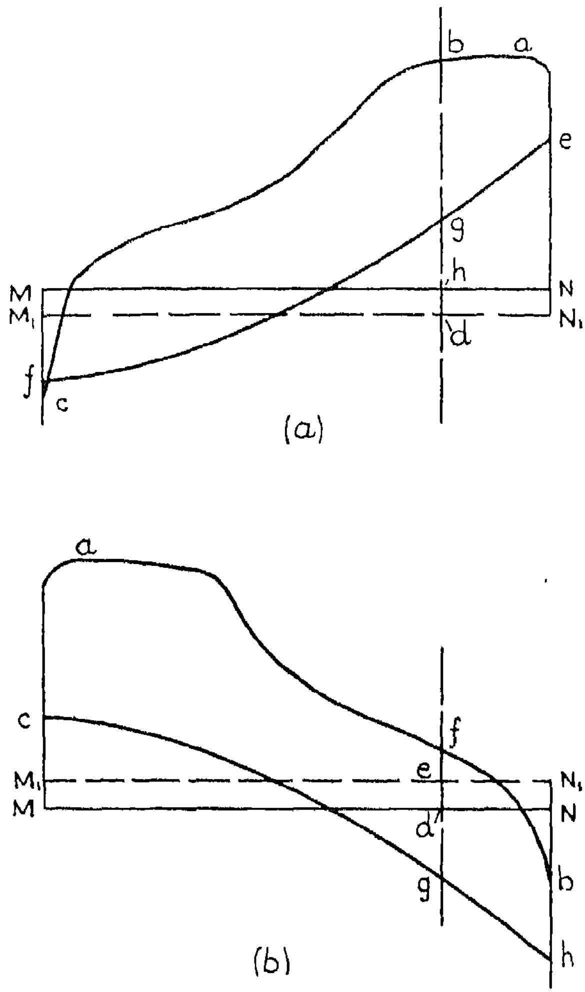


Fig. 131.

is negative. The curve of accelerating force is exactly similar to the acceleration diagram, Fig. 51, and we proceed to apply this correction to the effective-pressure diagram.

When $\theta = 0°$, accelerating force per square inch of piston area $= \frac{W}{gA}\omega^2 r\left(1+\frac{r}{l}\right)$. When $\theta = 180°$, accelerating force per square inch of piston area $= \frac{W}{gA}\omega^2 r\left(-1+\frac{r}{l}\right)$. In Fig. 131 (*a*) the ordinate $N_1 e$ is made proportional to $\frac{W}{gA}\omega^2 r\left(1+\frac{r}{l}\right)$ and $M_1 f$ proportional to $\frac{W}{gA}\omega^2 r\left(-1+\frac{r}{l}\right)$, the base line $M_1 N_1$ being the corrected base for a vertical engine. For a horizontal engine the ordinates are measured from N and M respectively. The curve *egf* is drawn to represent the curve of accelerating force per square inch of piston area for all positions of the piston on the outstroke. The diagram of available pressure per square inch is the net diagram *eabcfge*, and the ordinate *gb* is the net pressure per square inch for the given piston position.

For the instroke, Fig. 131 (*b*), the ordinate $M_1 c$ is made proportional to $\frac{W}{gA}\omega^2 r\left(-1+\frac{r}{l}\right)$ and the ordinate $N_1 h$ proportional to $\frac{W}{gA}\omega^2 r\left(1+\frac{r}{l}\right)$, and the curve *cgh* is the complete curve representing the accelerating force per square inch of piston area. The diagram of available pressure per square inch is *cafbhgc*, and for the given piston position the ordinate is *fg*. The total force available at the crosshead is the piston area A multiplied by the ordinate for the corresponding position of the piston.

§ **164. Turning-moment on Crank-shaft.** In Fig. 132, let A represent the crosshead, AB the connecting-rod, and BC the crank of a steam engine. The stroke is EF, and a diagram of available pressure per square inch is drawn immediately above EF, showing the variation of available pressure per square inch of piston area for the outstroke.

Let p = available pressure per square inch,

A = area of piston,

P = total available pressure at crosshead;

then $P = A \times p$,

turning-moment (from § 139) $= P \times CD$.

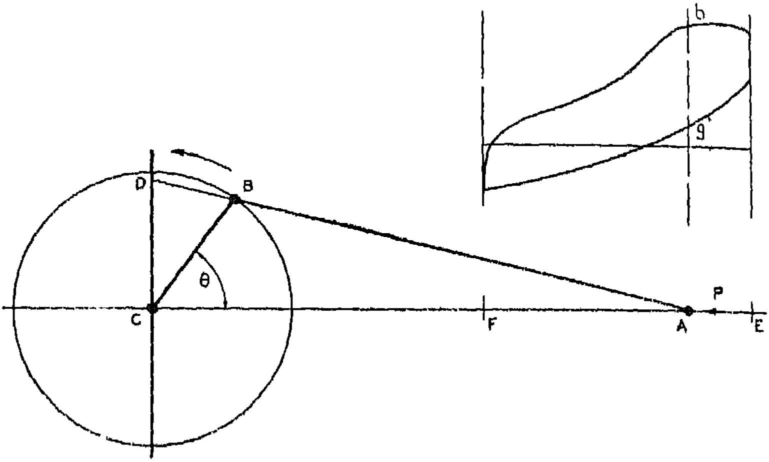


Fig. 132.

EXAMPLE 1. A double-acting vertical steam engine has the following particulars:

Diameter of cylinder	= 16 in.
Stroke	= 20 in.
Length of connecting-rod	= 45 in.
Revolutions per minute	= 80
Weight of reciprocating parts . . .	= 300 lb.

What is the effective pressure on the piston at the end of the down stroke, and its direction, if the force on the crank pin is zero at that instant? [*Inst. C. E.*]

Available force at crosshead = 0.

Available force = effective steam pressure on piston + weight of reciprocating parts — accelerating force.

0 = effective steam pressure

$$+300-\frac{300}{g}\cdot\left(\frac{80}{60}2\pi\right)^2\cdot\frac{10}{12}\left(-1+\frac{10}{45}\right).$$

∴ Effective steam pressure $= -300-424 = -724$ lb.

The effective steam pressure is -724 lb., the negative sign indicating that the pressure must act on the underside of the piston, or upwards. This corresponds to $\frac{724}{\frac{1}{4}\pi\times 16^2}$, or 3·6 lb. per sq in.

EXAMPLE 2. *It is found that when an engine crank has turned through an angle of 30° from the inner dead centre, the acceleration of the piston is 120 ft. per sec. per sec., the difference of steam pressure on the two sides of the piston is 70 lb. per sq. in., and the frictional resistance is equivalent to a force of 125 lb. If the area of the piston is 95 sq. in., the weight of the reciprocating parts 400 lb., the crank radius 11 in., and the length of the connecting-rod 48 in., determine the turning-moment on the crank-shaft at that instant.* [*Inst. C. E.*]

Assuming the engine to be horizontal,

$$\begin{aligned}\text{available force} &= \text{effective steam pressure}-\text{frictional resistance}-\text{accelerating force}\\ &= 70\times 95-125-\frac{400}{g}\times 120\\ &= 5{,}035 \text{ lb.}\end{aligned}$$

By construction (similar to Fig. 132) $CD = 6{\cdot}7$ in.,

$$\therefore \text{ turning-moment} = 5{,}035\times\frac{6{\cdot}7}{12} = 2{,}810 \text{ lb.-ft.}$$

For a vertical engine the available pressure is increased by 400 lb. and turning-moment $= 5{,}435\times\frac{6{\cdot}7}{12} = 3{,}030$ lb.-ft.

§ 165. Crank Effort. When the turning-moment on a crank-shaft is known for a given crank position, the effort required at crank radius and acting perpendicular to the crank to give the same turning-moment is known as the crank effort.

Let R = crank effort,

r = radius of crank.

From Fig. 132, turning-moment $= P\times CD = R\times r$. Since r is constant the crank effort is directly proportional to the magnitude of the turning-moment.

§ 166. Turning-moment Diagram. The method of finding the turning-moment on the crank-shaft for any piston

position or for the corresponding crank angle has already been described. If values of the turning-moment obtained in this manner are plotted against corresponding crank positions, the diagram thus obtained is known as a turning-moment diagram, and since the crank effort is directly proportional to the turning-moment, a turning-moment diagram is identical in shape with a crank-effort diagram; the scales to which the two diagrams are drawn will, of course, be different.

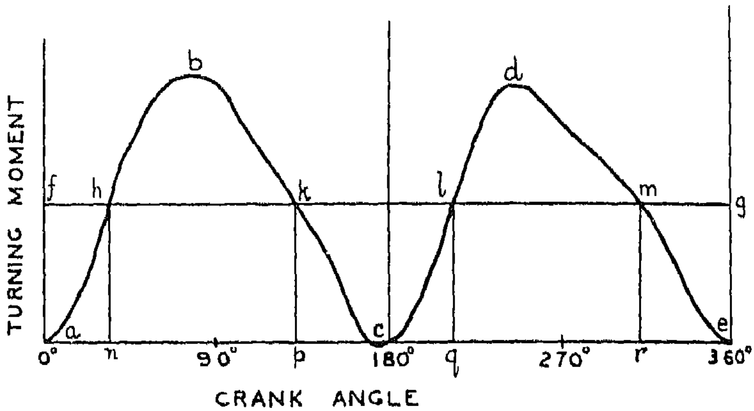


Fig. 133.

A turning-moment diagram for a single-cylinder double-acting steam engine is shown in Fig. 133. The horizontal ordinates represent crank angle and the vertical ordinates turning-moment. From the diagram it is seen that the turning-moment is zero when the crank angle is zero, rises to a maximum value when the crank angle is rather less than 90°, and is again zero at 180°. This part of the curve is represented by *abc* and is the turning-moment diagram for the outstroke. For the instroke the turning-moment diagram is somewhat similar, as shown at *cde*.

Since the product of a turning-moment and the angle turned through is the work done, the area of the turning-moment diagram is proportional to the work done per revolution. The mean resisting torque against which the engine is working is usually assumed constant and may be represented

by a horizontal line fg. The height of the ordinate af is the mean height of the turning-moment diagram, since it is assumed that the work done by the turning-moment per revolution is equal to the work done against the mean resisting torque per revolution, and the area of the rectangle $afge$ is proportional to the work done against the mean resisting torque.

§ 167. **Fluctuation of Energy.** The horizontal line fg cuts the turning-moment diagram in the points h, k, l, m. The ordinates hn, kp, lq, and mr are shown perpendicular to ae. When the crank moves from the position represented by n to position p, the work done by the engine is proportional to the area $nhbkp$, and the work done against the resisting torque is proportional to the area of the rectangle $nhkp$. The engine has thus done more work than has been taken from it, and this excess of work is represented by the area hbk. This excess work is stored in the fly-wheel and hence the speed of the fly-wheel increases while the crank moves from n to p.

Similarly, while the crank is moving from p to q more work is taken from the engine than is developed, and this loss of work is represented by the area kcl. To supply this loss, the fly-wheel gives up some of its energy and the speed decreases during this interval. As the crank moves from q to r, excess work is again developed proportional to the area ldm, and the speed again increases. As the crank moves from r to n (n being a corresponding position for the next revolution), loss of work, proportional to $meg+fah$, occurs, and the speed decreases. The areas hbk, kcl, ldm, etc., represent fluctuations of energy.

When the crank is at p, the engine has a maximum speed, since the fly-wheel has absorbed energy while the crank has moved from n to p. At q the engine has a minimum speed, since the fly-wheel has given out energy while the crank has moved from p to q. Similarly, at r and n the engine will have maximum and minimum speeds respectively. For the given turning-moment diagram there are thus two maximum speeds and two minimum speeds. The two maximum speeds are not necessarily equal. The speeds are maximum from the

mathematical point of view that the speed up to that point is increasing and after that point is decreasing. The greatest speed is the greater of the two maxima, and the least speed is the lesser of the two minima.

§ 168. Maximum Fluctuation of Energy. The turning-moment diagram for a single-cylinder engine varies considerably throughout the stroke. For engines with two cylinders, the resultant turning-moment diagram is the sum of the turning-moment diagrams for the two cylinders, and by having the cranks 90° apart the resultant turning-moment diagram has a less variation than that for a single cylinder. In

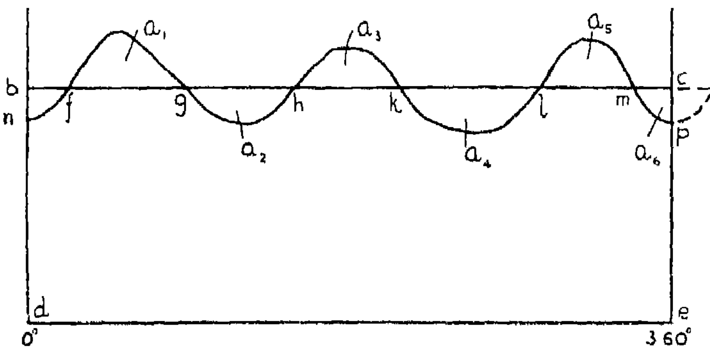


Fig. 134.

a three-cylinder engine with cranks at 120° the variation is still less.

In Fig. 134, let *bc* represent the constant resisting torque on the crank-shaft and the wavy line *nfghklmp* the turning-moment diagram for a given engine. The area under the wavy diagram is equal to the area of the rectangle *bced*. As already explained, when the crank is in positions corresponding to g, k, and m the speed of the engine at each of these positions is a maximum, and when the crank is in positions corresponding to f, h, and l the speed in each of these positions is a minimum.

Let a_1, a_3, and a_5 be the areas of the portions of the turning-moment diagram above *bc*, and a_2, a_4, and a_6 the areas of the portions below. Each of these areas represents some quantity

of energy which is either added to or taken away from the energy of the moving parts of the engine. The energy stored in a rotating fly-wheel is usually much greater than that stored in the other moving parts of an engine, and it is customary to assume that the fluctuations of energy (corresponding to the areas a_1, a_2, a_3, etc.) are absorbed by or added to the fly-wheel.

Let E = energy in the fly-wheel when the crank is in the position represented by f.

Then $E+a_1$ = energy in the fly-wheel for crank position g;

$E+a_1-a_2$ = energy in the fly-wheel for crank position h;

$E+a_1-a_2+a_3$ = energy in the fly-wheel for crank position k;

$E+a_1-a_2+a_3-a_4$ = energy in the fly-wheel for crank position l;

$E+a_1-a_2+a_3-a_4+a_5$ = energy in the fly-wheel for crank position m;

$E+a_1-a_2+a_3-a_4+a_5-a_6 = E$ = energy in the fly-wheel for crank position f.

The curve is continued to f in the next revolution, and since the area under the curve is equal to the area of the rectangle $bced$, $a_1-a_2+a_3-a_4+a_5-a_6 = 0$.

The greatest of these energies is the greatest energy stored in the fly-wheel, and for the crank position in which this occurs the engine has its greatest speed. The least of these energies is the least energy stored in the fly-wheel, and for the crank position in which this occurs the engine has its least speed. The difference between the kinetic energy in the fly-wheel at the greatest speed and the kinetic energy at the least speed is the *maximum fluctuation of energy*. The difference between the greatest speed and the least speed is the *maximum fluctuation of speed*. The ratio of the fluctuation of speed to the mean speed is the *coefficient of fluctuation of speed*.

Suppose the greatest of these energies occurs when the crank is at m and the least when the crank is at h.

$E+a_1-a_2+a_3-a_4+a_5$ = energy in fly-wheel for crank position m.

$E+a_1-a_2$ = energy in fly-wheel for crank position h.

Maximum fluctuation of energy

$$= (E+a_1-a_2+a_3-a_4+a_5)-(E+a_1-a_2)$$
$$= a_3-a_4+a_5.$$

The ratio of the maximum fluctuation of energy to the work done per cycle is the coefficient of the fluctuation of energy.

Work done per cycle = area under curve = area of rectangle *bced.*

$\therefore$ Coefficient of fluctuation of energy

$$= \frac{a_3-a_4+a_5}{\text{work done per cycle}} = \frac{a_3-a_4+a_5}{\text{area } bced}.$$

§ **169. Fly-wheels.** The rim of a fly-wheel is usually heavy in comparison with the boss and the spokes, and, since the radius of gyration of the rim is much greater than that of the boss and of the spokes, the moment of inertia of the rim is greatly in excess of the combined moment of inertia of the boss and spokes. In a preliminary estimation for the size of a fly-wheel it is usual, therefore, to neglect the effect of the boss and spokes and to assume that the whole of the energy of the fly-wheel is contained in the heavy rim. In many cases an allowance factor is used to take into account the boss and spokes.

Let E = kinetic energy of fly-wheel at mean speed,
W = weight of fly-wheel,
k = radius of gyration,
ω = mean speed in radians per second,
ω_1 = greatest speed,
ω_2 = least speed,
e = maximum fluctuation of energy.

$$E = \tfrac{1}{2}I\omega^2.$$

Fluctuation of kinetic energy in fly-wheel $= \frac{1}{2}I(\omega_1{}^2-\omega_2{}^2) = e$.
Since ω is the mean speed, $\omega = \frac{1}{2}(\omega_1+\omega_2)$.
Substituting for $\omega_1+\omega_2$,

$$e = I\omega(\omega_1-\omega_2).$$

Thus for a given maximum fluctuation of energy the moment of inertia I may be calculated for a given fluctuation of speed.

An alternative expression may be found incorporating the percentage fluctuation of speed.

Let q = total percentage fluctuation of speed;

then
$$q = \frac{\omega_1-\omega_2}{\omega}\times 100.$$

Now
$$e = \tfrac{1}{2}I(\omega_1-\omega_2)(\omega_1+\omega_2) = \tfrac{1}{2}I\frac{\omega q}{100}\times 2\omega = \tfrac{1}{2}I\omega^2\times\frac{2q}{100},$$

or
$$e = E\times\frac{2q}{100}.$$

EXAMPLE 3. The rim of an engine fly-wheel weighs 6·5 tons and the mean radius is 6 ft. It is found from a crank-effort diagram that the fluctuation of energy is 20 ft.-tons. Calculate the maximum speed of the wheel, and the minimum speed when the mean speed is 120 revs. per min. [*Inst. C. E.*]

$$e = I\omega(\omega_1-\omega_2),$$

$$20 = \frac{6\cdot5}{g}\times 6^2\times\frac{120}{60}\times 2\pi(\omega_1-\omega_2),$$

$$\therefore\ \omega_1-\omega_2 = 0\cdot219.$$

$$N_1-N_2 = 0\cdot219\times\frac{60}{2\pi} = 2\cdot09.$$

$$\text{Maximum speed} = 120+\frac{2\cdot09}{2} = 121\cdot04 \text{ revs. per min.}$$

$$\text{Minimum speed} = 120-\frac{2\cdot09}{2} = 118\cdot96 \text{ revs. per min.}$$

EXAMPLE 4. In a turning-moment diagram the areas above and below the mean torque line taken in order are 0·9, 0·5, 0·6, 0·8, 0·3, 0·6, 0·4, and 0·3 sq. in. respectively. The scales of the turning-moment diagram are

Turning-moment . . .	1 in. = 5,000 lb.-ft.
Crank angle	1 in. = 60°.

The mean speed of the engine is 120 revs. per min. and the variation of speed must not exceed ± 3 per cent. of the mean speed. Assuming the radius of gyration of the fly-wheel to be $3\frac{1}{2}$ ft., find the weight of fly-wheel to keep the speed within the given limits.

$$1 \text{ sq. in. on turning-moment diagram} = 5{,}000 \times 2\pi \times \frac{60}{360}$$
$$= 5{,}240 \text{ ft.-lb.}$$

Area representing the maximum fluctuation of energy

$$= +0{\cdot}4 - 0{\cdot}3 + 0{\cdot}9 - 0{\cdot}5 + 0{\cdot}6 = 1{\cdot}1 \text{ sq. in.}$$

$\therefore$ Maximum fluctuation of energy $= 5{,}240 \times 1{\cdot}1 = 5{,}760$ ft.-lb.

Total percentage fluctuation of speed $= 2 \times 3 = 6 = q$,

$$\therefore\ 5{,}760 = E \times \tfrac{12}{100},$$
$$E = 48{,}000 \text{ ft.-lb.}$$
$$E = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}.\frac{W}{g}(3\tfrac{1}{2})^2\left(\frac{120}{60} \times 2\pi\right)^2 = 48{,}000;$$
$$\therefore\ W = 1{,}600 \text{ lb.}$$

EXERCISES. X

1. A fly-wheel weighs 44·4 tons. Its radius of gyration is $11\frac{1}{2}$ ft. Find how much energy it stores at 60 revs. per min. If the variation of torque on the crank-shaft is such that 55 ft.-tons of energy are alternately added to and subtracted from the mean energy stored in the wheel, calculate the corresponding maximum and minimum number of revolutions of the wheel per minute. [*Inst. C. E.*]

2. The fly-wheel of an engine has a moment of inertia about the axis of the shaft of 8,000 ft.-lb. units. If the mean speed of the wheel is 50 revs. per min., calculate the fluctuation of energy per stroke if the maximum speed is at the rate of 52 and its minimum 48 revs. per min. [*Inst. C. E.*]

3. The following data refer to a single-cylinder vertical engine:

Diameter of cylinder	= 8 in.
Stroke	= 24 in.
Ratio $\dfrac{\text{connecting-rod}}{\text{crank}}$	$= 4\frac{1}{2}$
Revolutions per minute . . .	= 75
Weight of reciprocating parts . .	= 50 lb.

Find, by graphical or other means, the turning effort when the crank has turned through an angle of 135° from the inner dead centre, assuming that the effective steam pressure on the piston is 25 lb. per sq. in. [*Inst. C. E.*]

4. Explain how the turning-moment diagram for a steam engine is used to determine the weight of fly-wheel necessary. Such a diagram is drawn to scales of 1 in. = 5,000 ft.-lb. and 1 in. = 50°, and the maximum energy fluctuation is shown by an area of 0·95 square in. The mean speed of revolution is 130 per min., and the mean rim velocity of the fly-wheel is to be about 80 ft. per sec. If the coefficient

of speed fluctuation allowable is 0·02, determine the necessary weight of the fly-wheel rim if the energy in the arms is $\frac{1}{13}$th of the energy in the rim. [*I. Mech. E.*]

5. A fly-wheel when running at 90 revs. per min. has a stored energy of 3,000,000 ft.-lb. By reason of additional load it is slowed down to 86 revs. per min. in two seconds. By how much will the stored energy be reduced, and what is the average horse-power produced by the slowing-down of the fly-wheel? [*Inst. C. E.*]

6. Find the force acting on the piston rod at each end of the stroke in a double-acting horizontal steam engine when running under the following conditions:

Weight of piston and rod	= 275 lb.
Radius of crank	= 15 in.
Length of connecting-rod	= 60 in.
Revolutions per minute	= 150
Steam pressure—constant throughout stroke	= 100 lb. sq. in.
Back " " " "	= 3 lb. sq. in.
Diameter of cylinder	= 16 in.

[*Inst. C. E.*]

7. The reciprocating parts of a steam engine weigh $8\frac{1}{8}$ tons; the connecting-rod is $17\frac{1}{2}$ ft. long and the crank radius is 3 ft. Calculate the force required to accelerate the reciprocating masses at the beginning and end of the stroke when the speed is 60 revs. per min. [*Inst. C. E.*]

8. In a horizontal engine the effective pressure on the piston is 55 lb. per sq. in. and the acceleration is 110 ft. per sec. per sec. when the crank has turned through an angle of 30° from the inner dead centre. The friction of the reciprocating parts is equivalent to a force of 200 lb. The piston diameter is 9 in., crank radius 10 in., weight of reciprocating parts 350 lb., and length of connecting-rod 40 in. Determine the turning-moment on the crank-shaft for this position of the crank.

9. The rim of an engine fly-wheel weighs 7 tons and the mean radius is 6 ft. This fly-wheel is on a crank-shaft, and it is found that the maximum fluctuation of energy is 22 ft.-tons. If the mean speed is 120 revs. per min., find the greatest and least speeds.

10. The crank and connecting-rod of an engine are 1 ft. and 4 ft. long respectively. When the crank has turned through an angle of 30° from the inner dead centre, the pressure on the piston is 20,000 lb. There is a fly-wheel weighing 2 tons and having a radius of gyration of 3 ft. fixed to the crank-shaft. Neglecting all resistances and inertia of all parts except the fly-wheel, find: (*a*) the thrust along the connecting-rod; (*b*) the turning effect in the crank-shaft; (*c*) the angular acceleration of the fly-wheel. [*I. Mech. E.*]

11. A gas engine has a stroke of 10 in. and a piston diameter of 6 in., and a ratio of connecting-rod to crank of $4\frac{1}{2}$ to 1. Find the acceleration

of the piston at three-quarters of the compression stroke and at dead centre at the end of that stroke if the engine is running at 400 revs. per min.

Assuming the weight of the accelerated parts to be 35 lb., find the force required to give the piston its necessary acceleration when at dead centre at the end of the compression stroke, and find the minimum pressure in lb. per sq. in. of the gas in the cylinder in order that this force shall be provided by the compressed charge. [*I. Mech. E.*]

12. A steel fly-wheel, 4 ft. outside diameter, has a rim 4 in. sq. in section, a web 1 in. thick connecting the rim to the boss, and a boss 8 in. diameter and 8 in. long. Determine: (*a*) the moment of inertia of the wheel; (*b*) the kinetic energy of the wheel when rotating at 300 revs. per min.

Weight of 1 cu. in. of steel = 0·28 lb. [*I. Mech. E.*]

13. The crank of a steam engine is 1 ft. radius, and the connecting-rod is 4 ft. long. The fly-wheel weighs 4 tons, and the rim has a mean radius of 4 ft. (The mass of the fly-wheel may be assumed concentrated at the mean radius.)

When the crank makes 45° with the position at the inner dead centre, the pressure on the piston is 25,000 lb.

The resisting torque at this instant is 17,000 lb.-ft.

Find the angular acceleration of the fly-wheel in sign and magnitude. [*I. Mech. E.*]

14. The fluctuation of energy of a double-acting engine of 100 horse-power when making 100 revs. per min. is ± 2 per cent. of the energy developed per stroke. Find the weight of fly-wheel, of 6 ft. diameter, necessary to keep down the fluctuation of speed within ± 1 per cent.

15. The fly-wheel of a steam engine has a mass of 50 tons, and its radius of gyration is 12 ft. The other moving parts are comparatively light.

When the engine is running unloaded at 52 revs. per min., the steam is shut off suddenly, and the engine comes to rest in 4 min. 10 sec.

Assume that the frictional resistances are constant during this period, and find the horse-power required to drive the engine when running unloaded at 52 revs. per min. [*Lond. B.Sc.*]

16. The turning-moment diagram for a four-stroke gas engine may be assumed for simplicity to be represented by four triangles, the areas of which measured from the line of zero pressure are as follows:

Expansion stroke	. . .	= 8·50 sq. in.
Exhaust ,,	. . . .	= 0·80 ,,
Suction ,,	. . . .	= 0·56 ,,
Compression ,,	. . .	= 2·14 ,,

Each square inch represents 1,000 lb.-ft.

Assuming the resisting torque to be uniform, find the weight of

the rim of a fly-wheel required to keep the speed between 98 and 102 revs. per min.; the mean radius of the rim is 3 ft. [*Lond. B.Sc.*]

17. The centre line of the cylinder of a single-acting horizontal steam engine is designed to be 6 in. below the axis of the crank-shaft. The crank is 6 in. long, the connecting-rod 18 in., and the piston diameter 7 in.

The steam pressure is constant throughout the working stroke at 80 lb. pressure by gauge, and the exhaust is direct to the atmosphere with no compression at the end.

If the engine runs at 180 revs. per min. and is completely balanced, find the moment of inertia of the fly-wheel necessary to limit the fluctuation of speed to $1\frac{1}{2}$ per cent. of the mean speed. [*Lond. B.Sc.*]

18. Write a short essay on the method of finding the weight of fly-wheel for a single-cylinder gas engine so as to maintain the speed fluctuation within given limits. Take into account the inertia of the moving parts. [*Lond. B.Sc.*]

19. A double-acting steam engine develops 48 horse-power; the stroke of the piston is 14 in., and the crank makes 120 revs. per min. The fly-wheel has a mean diameter of 5 ft. 8 in.

The maximum fluctuation of energy during a revolution is 30 per cent. of the work done per stroke.

Determine the weight of the fly-wheel rim if the total fluctuation of velocity is not to be more than 2 per cent. of the mean velocity. Prove the formula you use. [*Lond. B.Sc.*]

20. A high speed vertical inverted engine runs at 450 revs. per min., the stroke is 8 in., the reciprocating parts weigh 0·93 lb. per sq. in. of piston area, and the connecting-rod is 16 in. long. Determine the acceleration pressure per square inch of piston area at each end of the stroke.

Show by a sketch how the acceleration pressure varies during one stroke. [*Lond. B.Sc.*]

21. An engine developing 1,000 horse-power at 240 revs. per min. has a wire-wound fly-wheel weighing 60 tons and having a radius of gyration of 9 ft. Express the energy stored in the fly-wheel in terms of the work done by the engine per revolution. If steam were shut off, what moment of resistance would reduce the speed from 200 to 100 revs. per min. in $1\frac{1}{2}$ min? [*Lond. B.Sc.*]

22. The following particulars are taken from a horizontal steam engine:

Revolutions per minute . . .	= 250
Length of connecting-rod . .	= 7 ft.
Length of stroke	= 26 in.
Diameter of cylinder . . .	= 19 in.
Weight of reciprocating parts . .	= 550 lb.

When the crank angle is at an angle of 30° after the inner dead centre, the net steam pressure per square inch on the piston is 85 lb.

Determine, for this position of the crank, the thrust along the connecting-rod, the turning-moment on the crank-shaft, and the rate at which work is being done on the crank-shaft. [*Lond. B.Sc.*]

23. In the turning-moment diagram for one revolution of a steam engine the areas above and below the curve of resistance, taken in order, are $+0{\cdot}53$, $-0{\cdot}33$, $+0{\cdot}38$, $-0{\cdot}47$, $+0{\cdot}18$, $-0{\cdot}36$, $+0{\cdot}35$, and $-0{\cdot}28$ sq. in. The scales of the diagram are:

Turning-moment . . . 1 in. = 8,000 lb.-ft.
Crank angle . . . . 1 in. = 60°.

The mean revolutions per minute are 150, and the total fluctuation of speed must not exceed 3 per cent. of the mean. Determine a suitable cross-sectional area of the rim of the fly-wheel, assuming the total energy of the fly-wheel to be $\frac{19}{18}$ that of the rim. The peripheral velocity of the fly-wheel is to be 50 ft. per sec. Take the weight of the material as 0·25 lb. per cu. in. [*Lond. B.Sc.*]

24. Find the weight of fly-wheel required for a steam engine in which it is desired to limit the cyclical variation of speed to 0·6 per cent. on either side of the mean. The ratio of the excess energy during any one stroke to the total work done per stroke is 0·28. The engine is designed to indicate 480 horse-power at 110 revs. per min. The radius of gyration of the wheel is 6·4 ft. [*Lond. B.Sc.*]

25. A horizontal steam-engine cylinder is 20 in. diameter, stroke 18 in., connecting-rod $3\frac{1}{2}$ ft. between centres, mass of reciprocating parts 280 lb. Find how much the pressure per square inch of cushion steam must exceed that on the other side of the piston, at each end of the stroke, so as to relieve the crank-pin brasses of all pressure when the engine is running at 350 revs. per min.

26. A steam engine develops 80 indicated horse-power at 100 revs. per min. against a steady load. The fly-wheel weighs 3 tons and has a radius of gyration of 5 ft. If the load suddenly changes to $\frac{1}{8}$ of the initial value, and the steam supply does not change for the next two revolutions, calculate the change of speed during this period.

27. The indicated horse-power of an engine is 100; mean speed 200 revs. per min. The energy to be absorbed by the fly-wheel is 10 per cent. of the work done in the cylinder per revolution. If the radius of gyration of the fly-wheel is 30 in., determine its weight in order that the total fluctuation of speed may not exceed 2 per cent. of the mean speed.

28. Explain the functions of a fly-wheel, and show how its size and mass may be calculated by the aid of a turning-moment diagram.

An engine develops 400 indicated horse-power at 120 revs. per min., and the energy fluctuation is shown by the turning-moment diagram to be 0·12 of the work done per revolution. Calculate the mass of the fly-wheel necessary to keep the speed fluctuation within 1·5 per cent. of the mean speed. Assume the radius of gyration of the wheel to be 5 ft. [*I. Mech. E.*]

CHAPTER XI

GOVERNORS

§ 170. **Function of a Governor.** The function of a governor as applied to engines is to keep the speed of the engine within prescribed limits for variations of load upon the engine. The speed of an engine may vary during each revolution, and *this variation is known as a cyclical variation*; *also it may* vary due to a variation of load upon the engine or due to a variation of the pressure or quantity of steam supply in the case of a steam engine. The regulation of speed in the former case is accomplished by the use of a suitable fly-wheel. If the boiler is functioning correctly, the quantity and pressure of steam will be maintained, and any variation of speed will be due to an alteration of the load; the governor thus regulates the speed of the engine, within certain limits, for alterations of load upon the engine. The regulation of the speed by the governor is accomplished by automatically regulating the power developed by the engine so that it is approximately equal to the power taken from the engine. The operation of the fly-wheel is continuous, whereas the operation of the governor is usually more or less intermittent.

For steam engines the governor may regulate the power developed in two ways: (1) by reducing the pressure of the steam before it is admitted to the cylinder—this method is known as throttling, and the governor is called a throttle valve governor; (2) by varying the quantity of steam admitted to the cylinder per stroke—usually by manipulation of the valve so as to alter the point of cut-off. This latter type is obviously more economical since the steam is always used at the supply pressure. Throttle valve governors are limited in their application in that they can reduce the pressure of steam but cannot increase the pressure.

For gas and oil engines the regulation of the speed is usually accomplished by the control of the amount of gas or oil admitted per cycle. Another method is that of varying the ratio of air and gas or oil as the case may be.

§ 171. Action of Simple Governor. A simple governor of the Watt type is shown in Fig. 135. A vertical spindle S is driven through gearing from the crank-shaft of the engine, so that the spindle always rotates either at the same speed as the crank-shaft or some definite fraction of the speed. At the top

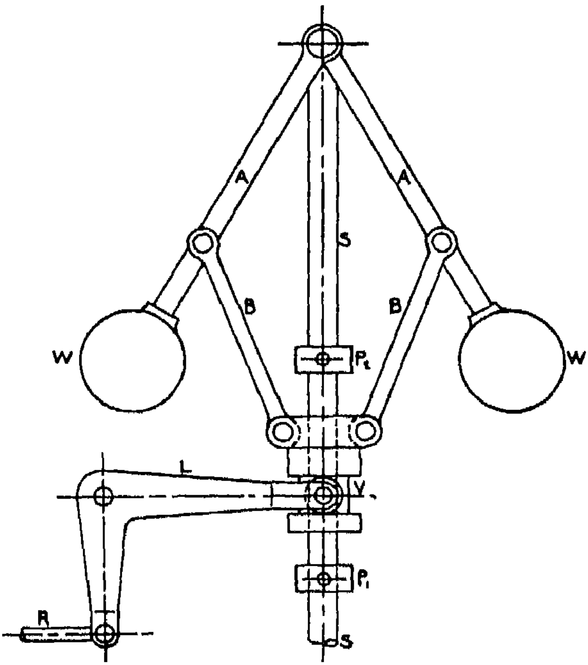


Fig. 135.

of the vertical spindle S two swinging arms A are pinned, and at the free extremity of each arm is a heavy mass W. The arms rotate with the spindle. A sleeve V is connected to the arms A by links B. The sleeve V is free to slide up and down the spindle S but rotates with the spindle. Stops P_1 and P_2 fixed to the spindle limit the movement of the sleeve. A bell crank lever L is forked at one end and partially embraces the recess in the sleeve; the other end of the bell crank lever is pinned to a rod R which can operate, say, a

throttle valve. A vertical movement of the sleeve moves the rod R which actuates the regulating gear.

When a mass is rotating in a circular path there is a centripetal acceleration of $\omega^2 r$ towards the centre of rotation, and if the weight of the mass is W the centripetal or radial force acting towards the centre to cause this acceleration is $\frac{W}{g}\omega^2 r$. In the simple Watt governor shown in Fig. 135, this force is supplied by the weight of the sleeve, the weight of the rotating mass, and the resistance of the regulating gear. Acting outwards on each of the masses W is a centrifugal force of $\frac{W}{g}\omega^2 r$ lb. These centrifugal forces balance the radial inward force due to the weight of the sleeve, etc. If the speed of the engine increases, due to a reduction of load, the centrifugal force increases, and if the weight of the sleeve, etc., is insufficient to balance this increased force the masses W move out to a larger radius. This causes the sleeve to rise on the spindle, thereby actuating the rod R which is instrumental in decreasing the power of the engine to meet the decreased load. If the load on the engine is increased, with a consequent reduction of speed, the centrifugal force is less and the weight of the sleeve, etc., is in excess of this force. The sleeve thus falls on the spindle, again actuating the rod R and regulating the power to suit the increase of load.

§ **172. Simple Conical Governor.** The simplest form of conical governor consists of two rotating masses diametrically opposed. In Fig. 136 (*a*), (*b*), and (*c*), one rotating mass is shown attached to the end of an arm which rotates with the vertical spindle. The mass is connected directly to the sleeve by another arm which is not shown. The upper arm may be suspended from the vertical spindle in three ways: (1) from the axis of the spindle, (2) from a point attached to a collar on the spindle so that the arm produced intersects the spindle, (3) from a point attached to a collar so that the arm crosses the spindle. The three methods are shown at (*a*), (*b*), and (*c*). The *height* of the governor is the distance from the centre of the mass to the point of intersection between the arm and the

axis of the spindle. The arms are usually light in comparison with the rotating masses, and it is usual to neglect the effect of the arms when considering the forces acting on the masses.

For a preliminary investigation let the weight of the sleeve and resistance of regulating gear be neglected. Then the

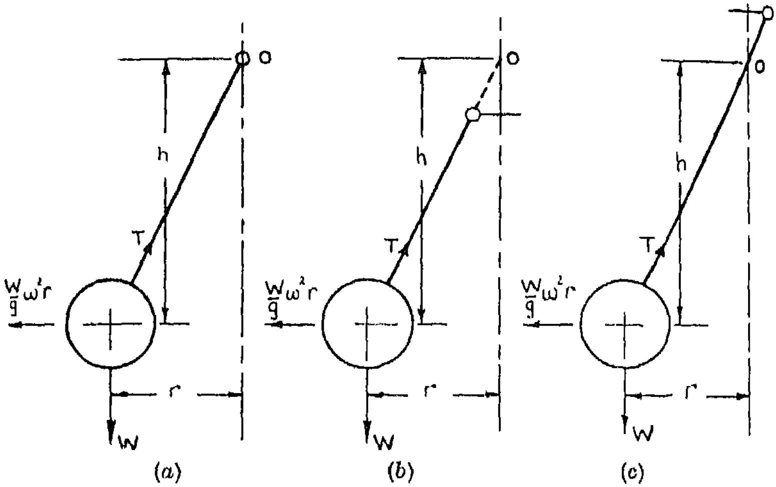


Fig. 136.

forces acting on each mass are, W acting vertically downwards, the centrifugal force $\frac{W}{g}\omega^2 r$ acting radially outwards, and the tension in the arm.

Let W = weight of each rotating mass,
r = radius of circular path,
h = height of governor,
ω = angular speed,
T = tension in upper arm.

Let O be the point of intersection of the arm and the axis of the spindle, then taking moments about O for all three cases,

$$\frac{W}{g}\omega^2 r \times h = W \times r,$$

$$\therefore\ h = \frac{g}{\omega^2}.$$

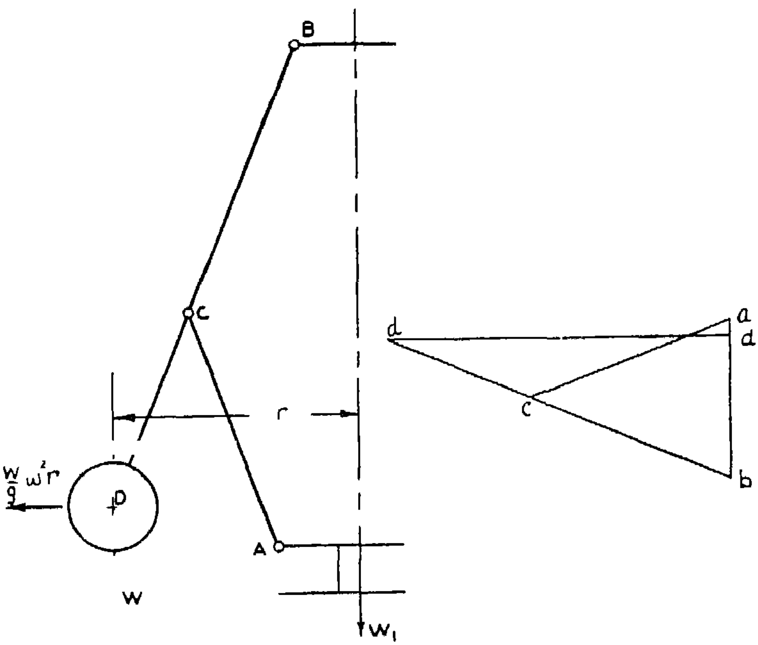
The height h is thus independent of W and varies inversely

as the square of the speed. Take a simple governor of this type rotating at 150 revs. per min.

$$\omega = \frac{150}{60} \times 2\pi,$$

$$h = \frac{32 \cdot 2}{25\pi^2} = 0 \cdot 1305 \text{ ft.} = 1 \cdot 566 \text{ in.}$$

For a height of 1·566 in. the arms are nearly horizontal and any alteration of speed would only cause a very small alteration of h, with a consequent movement of the sleeve of about twice this quantity, since for equal upper and lower arms the sleeve has twice the vertical movement of the mass W. This small movement of the sleeve would be insufficient to affect the alteration of power and consequently this type of governor is unsuitable, even for moderate speeds.



(a) Fig. 137. (b)

§ 173. Graphical Solution. In the form of Watt governor shown in Fig. 137 (*a*), *A* is a point on the sleeve and *B* is

a point on a collar fixed to the vertical spindle. Governor problems of this nature may be conveniently solved by applying the principle of work, and to apply the principle of work it is necessary to know the relative velocities or displacements which take place between the various parts of the mechanism. The rate at which work is done by the centrifugal force is the magnitude of the centrifugal force multiplied by its velocity in the direction of that force, and this work is balanced by the rate at which work is done in lifting the rotating masses, sleeve, etc.

Let W_1 = weight of sleeve, the frictional resistance of the sleeve and regulating gear being neglected.

The relative velocities between points on the mechanism are obtained by drawing a velocity diagram as in Fig. 137 (*b*).

Velocity of A relative to B = velocity of A relative to C + velocity of C relative to B.

$$ba = ca + bc = bc + ca.$$

Assuming the sleeve is moving upwards, the velocity of A relative to B is parallel to the vertical spindle and ba is drawn vertical, of any convenient length, to represent this velocity. The velocity of A relative to C is perpendicular to AC, and the vector ac is thus drawn perpendicular to AC. The velocity of C relative to B is perpendicular to CB, and the vector bc is drawn perpendicular to CB. The intersection of ac and bc locates c. The point d is located by producing bc to d such that $\frac{bc}{bd} = \frac{BC}{BD}$. The vector bd represents the velocity of D relative to B. If dd_1 is drawn horizontal, then $d_1 d$ is the *horizontal* component of the velocity of D relative to B, and bd_1 is the *vertical* component.

Work done by centrifugal force $= \frac{W}{g}\omega^2 r \times d_1 d$.

Work done in lifting the mass $W = W \times$ vertical velocity $= W \times bd_1$.

Work done in lifting the sleeve

$= \frac{1}{2}W_1 \times$ vertical velocity of sleeve

$= \frac{1}{2}W_1 \times ba$.

The weight of sleeve per mass W is $\frac{1}{2}W_1$.

Equating the work done by the centrifugal force to the work done in lifting the rotating mass and the sleeve,

$$\frac{W}{g}\omega^2 r \times d_1 d = W \times bd_1 + \tfrac{1}{2}W_1 \times ba.$$

From this equation the weight of the mass W_1 can be calculated for a given radius r, a given speed ω, and a given rotating mass W.

§ **174. The Friction at the Sleeve.** The resistance of the sleeve and regulating gear affects the speed of the governor for a given mass W. This resistance is usually expressed as a friction force at the sleeve, i.e. the force required to overcome the friction of the sleeve, and the resistance of the regulating gear may be expressed as an equivalent force acting at the sleeve. Since friction always opposes motion, the friction may be regarded as an additional load at the sleeve when the speed is increasing and the sleeve is being lifted. When the speed is decreasing, the friction force opposes the motion of the sleeve, which is downwards, and may be regarded as a reduction in the weight of the sleeve. In Fig. 137 let f be the equivalent friction at the sleeve, i.e. $\frac{1}{2}f$ for each rotating mass.

When the speed is increasing, the work done in overcoming friction is $+\frac{1}{2}f \times$ velocity of sleeve $= +\frac{1}{2}f \times ba$.

When the speed is decreasing, the work done in overcoming friction is $-\frac{1}{2}f \times ba$.

These two expressions for friction may be combined as one, and work done in overcoming friction at the sleeve $= \pm\frac{1}{2}f \times ba$, the positive sign to be used when the speed is increasing and the negative sign when the speed is decreasing.

The complete equation for equilibrium is

$$\frac{W}{g}\omega^2 r \times d_1 d = W \times bd_1 + \tfrac{1}{2}W_1 \times ba \pm \tfrac{1}{2}f \times ba.$$

For a given rotating mass W and a given radius r there are thus two speeds of the governor, and between these speeds the governor will not alter its configuration, and hence will not function.

EXAMPLE 1. In a Watt governor similar to that shown in Fig. 137 (*a*), B is $1\frac{1}{2}$ in. and A is 2 in. from the axis of the vertical spindle, AC is 6 in., BC 7 in., and BD 12 in. Find the speeds of the governor for the given configuration:

(*a*) neglecting the weight of the sleeve;

(*b*) if the sleeve weighs 2 lb.;

(*c*) if the friction at the sleeve is $1\frac{1}{2}$ lb.

The rotating mass W weighs 6 lb. and its radius is 6 in.

The velocity diagram is drawn in Fig. 137 (*b*) and on the original diagram $ba = 1{\cdot}23$ in., $bd_1 = 1{\cdot}1$ in., and $d_1 d = 2{\cdot}69$ in.

(*a*)
$$\frac{W}{g}\omega^2 r \times d_1 d = W \times bd_1.$$

$$\omega^2 = \frac{1{\cdot}1 \times 32{\cdot}2}{\frac{6}{12} \times 2{\cdot}69} = 26{\cdot}33,$$

$$\therefore \ \omega = 5{\cdot}14 \text{ radians per sec.}$$

and
$$N = 5{\cdot}14 \times \frac{60}{2\pi} = 49 \text{ revs. per min.}$$

(*b*)
$$\frac{W}{g}\omega^2 r \times d_1 d = W \times bd_1 + \tfrac{1}{2} W_1 \times ba.$$

$$\omega^2 = \frac{(6 \times 1{\cdot}1) + \frac{2}{2} \times 1{\cdot}23}{6 \times \frac{6}{12} \times 2{\cdot}69} \times 32{\cdot}2 = 31{\cdot}25,$$

$$\therefore \ \omega = 5{\cdot}59 \text{ radians per sec.}$$

and
$$N = 5{\cdot}59 \times \frac{60}{2\pi} = 53{\cdot}4 \text{ revs. per min.}$$

(*c*)
$$\frac{W}{g}\omega^2 r \times d_1 d = W \times bd_1 + \tfrac{1}{2} W_1 \times ba \pm \tfrac{1}{2} f . ba.$$

$$\omega^2 = \frac{(6 \times 1{\cdot}1) + (\frac{2}{2} \times 1{\cdot}23) \pm \left(\frac{1\frac{1}{2}}{2} \times 1{\cdot}23\right)}{6 \times \frac{6}{12} \times 2{\cdot}69} \times g = \frac{7{\cdot}83 \pm 0{\cdot}923}{8{\cdot}07} \times 32{\cdot}2.$$

Let ω_1 = maximum speed,

ω_2 = minimum speed,

$$\omega_1^2 = \frac{7{\cdot}83 + 0{\cdot}923}{8{\cdot}07} \times 32{\cdot}2 = 34{\cdot}9.$$

$$\therefore \ \omega_1 = 5{\cdot}91 \text{ radians per sec.}$$

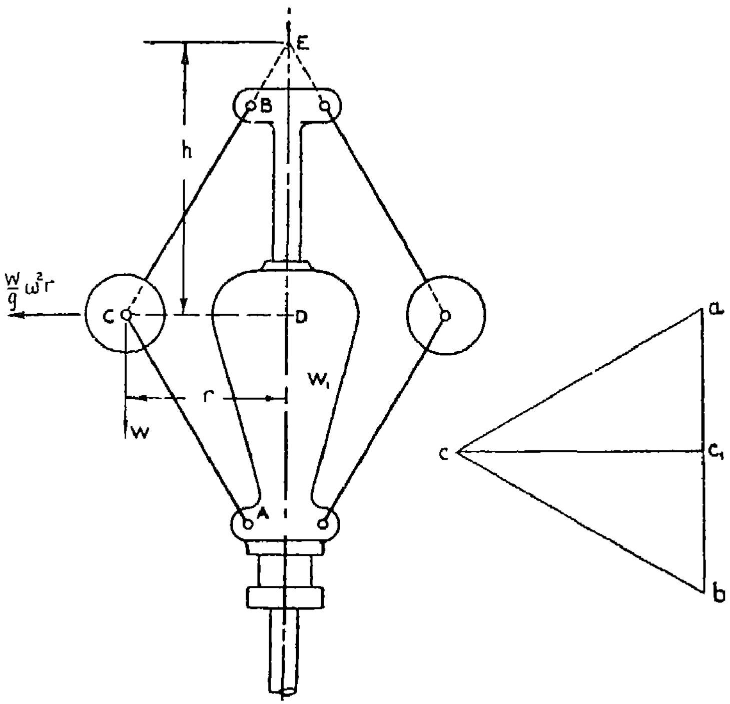
$$N_1 = 5{\cdot}91 \times \frac{60}{2\pi} = 56{\cdot}5 \text{ revs. per min.}$$

$$\omega_2^2 = \frac{7{\cdot}83 - 0{\cdot}923}{8{\cdot}07} \times 32{\cdot}2 = 27{\cdot}55,$$

$$\omega_2 = 5{\cdot}25 \text{ radians per sec.}$$

$$N_2 = 5{\cdot}25 \times \frac{60}{2\pi} = 50{\cdot}1 \text{ revs. per min.}$$

§ **175. Porter Governor.** The addition of a central weight or load to a governor of the Watt type increases the speed at which it may be run. The rotating masses W are suspended



(a) Fig. 138. (b)

by links such as BC (Fig. 138 (a)) and are attached to the sleeve by links such as CA. The central load W_1 is in one piece with the sleeve.

The velocity diagram is shown in Fig. 138 (b), in which ba is drawn of any convenient length parallel to the vertical spindle, bc is perpendicular to BC, and ac perpendicular to AC. The line cc_1 is drawn perpendicular to ab. c_1c represents the horizontal velocity of the rotating mass W, bc_1 the vertical velocity, and ba the vertical velocity of the sleeve.

Equating the work done by the centrifugal force in the direction of its motion to the work done in lifting the mass and the central load W_1,

$$\frac{W}{g}\omega^2 r \times c_1 c = W \times bc_1 + \tfrac{1}{2} W_1 \times ba,$$

or

$$\frac{W}{g}\omega^2 r \cdot \frac{c_1 c}{bc_1} = W + \tfrac{1}{2} W_1 \cdot \frac{ba}{bc_1}.$$

In the usual type of Porter governor, the arms BC and CA are equal and B and A are at equal distances from the axis of the vertical spindle. In this case $bc_1 = c_1 a$ and $ba = 2bc_1$. The triangles CDE and $bc_1 c$ are similar, and

$$\frac{c_1 c}{bc_1} = \frac{ED}{CD}, \quad \text{also} \quad \frac{ba}{bc_1} = 2,$$

$$\therefore \frac{W}{g}\omega^2 r \cdot \frac{ED}{CD} = W + \tfrac{1}{2} W_1 \times 2:$$

$$\text{but} \quad CD = r \quad \text{and} \quad DE = h,$$

$$\therefore \frac{W}{g}\omega^2 r \frac{h}{r} = W + W_1,$$

or

$$\omega^2 = \frac{W + W_1}{W} \cdot \frac{g}{h}.$$

In the more general case where the arms are not equal, let $\dfrac{ba}{bc_1} = q =$ ratio of vertical movement of sleeve to vertical movement of the rotating mass. Then

$$\frac{W}{g}\omega^2 r \times \frac{c_1 c}{bc_1} = W + \tfrac{1}{2} W_1 \times \frac{ba}{bc_1}.$$

But

$$\frac{c_1 c}{bc_1} = \frac{DE}{CD} = \frac{h}{r},$$

$$\therefore \frac{W}{g}\omega^2 r \times \frac{h}{r} = W + \tfrac{1}{2} W_1 \cdot q,$$

$$\omega^2 = \frac{W + \tfrac{1}{2} W_1 q}{W} \cdot \frac{g}{h}.$$

The effect of friction at the sleeve is readily taken into account as before.

Let f = friction at the sleeve:

$$\omega^2 = \frac{W+\frac{1}{2}W_1 q \pm \frac{1}{2}fq}{W} \cdot \frac{g}{h}.$$

EXAMPLE 2. A loaded governor of the Porter type has equal links 10 in. long pivoted at the axis; the weight of each ball is 6 lb. and the weight of the central load is 28 lb. The ball radius is 6 in. when the governor begins to lift and 8 in. when at maximum speed. Determine the maximum speed and range of speed. [*Inst. C. E.*]

For maximum speed $r = 8$ in., $\therefore\ h = \sqrt{(10^2-8^2)} = 6$ in.

$$\omega_1^2 = \frac{W+\frac{1}{2}W_1 q}{W} \cdot \frac{g}{h} = \frac{6+14\times 2}{6} \times \frac{32 \cdot 2}{0 \cdot 5} = 365.$$

$$\therefore\ \omega_1 = 19 \cdot 1 \text{ radians per sec.,}$$

$$N_1 = 19 \cdot 1 \times \frac{60}{2\pi} = 182 \cdot 5 \text{ revs. per min}$$

When the governor begins to lift, $r = 6$ in.,

$$\therefore\ h = \sqrt{(10^2-6^2)} = 8 \text{ in.}$$

$$\omega_2^2 = \frac{6+28}{6} \times \frac{32 \cdot 2}{\frac{8}{12}} = 273 \cdot 5,$$

$$\therefore\ \omega_2 = 16 \cdot 53 \text{ radians per sec.,}$$

$$N_2 = 158 \text{ revs. per min.}$$

Range of speed $= 182 \cdot 5 - 158 = 24 \cdot 5$ revs. per min.

EXAMPLE 3. If the friction at the sleeve of the Porter governor in Example 2 is equivalent to 3 lb., find the maximum and minimum speeds and the range of speed.

For maximum speed, $h = 6$ in., and

$$\omega_3^2 = \frac{6+28+3}{6} \times \frac{32 \cdot 2}{0 \cdot 5} = 297,$$

$$\omega_3 = 19 \cdot 93,$$

$$N_3 = 190 \cdot 6 \text{ revs. per min.}$$

For minimum speed, $h = 8$ in., and

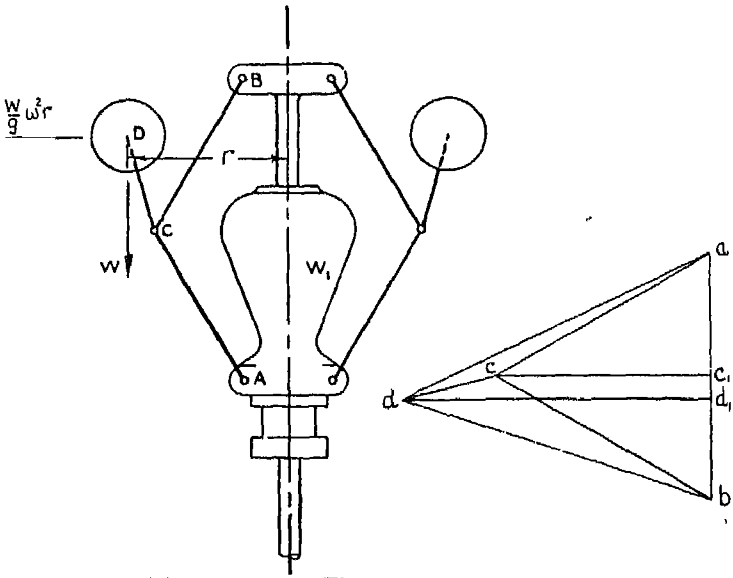
$$\omega_4^2 = \frac{6+28-3}{6} \times \frac{32 \cdot 2}{\frac{8}{12}} = 249 \cdot 5,$$

$$\omega_4 = 15 \cdot 8,$$

$$N_4 = 151 \text{ revs. per min.}$$

Range of speed $= 190 \cdot 6 - 151 = 39 \cdot 6$ revs. per min.

§ **176. Proell Governor.** Another form of governor known as the Proell governor is shown in Fig. 139 (*a*). The lower link AC is continued to D at the end of which the rotating mass is carried. The link DCA is in one piece, bent at C. In the usual type of Proell governor the arms AC and BC are equal and the points A and B are at equal distances



(*a*) Fig. 139. (*b*)

from the axis of the vertical spindle. The velocity diagram is shown in Fig. 139 (*b*), in which abc is constructed as for the Porter governor. The velocity of D relative to A is perpendicular to a line joining A and D, and the velocity of D relative to C is perpendicular to CD. The vectors ad and cd are accordingly drawn perpendicular to AD and CD respectively. Joining bd gives the velocity of D relative to B. dd_1 is drawn perpendicular to ab, and bd_1 represents the vertical velocity of the rotating mass W and $d_1 d$ the horizontal velocity.

Equating the work done, as before,

$$\frac{W}{g}\omega^2 r \times d_1 d = W \times bd_1 + \tfrac{1}{2} W_1 \times ba.$$

If the friction at the sleeve is equivalent to f lb.,

$$\frac{W}{g}\omega^2 r \times d_1 d = W \times bd_1 + \tfrac{1}{2}W_1 \times ba \pm \tfrac{1}{2}f \times ba.$$

§ **177. Alternative Method of Solution.** Problems in connexion with governors of the Porter and Proell type may be solved alternatively by considering the forces acting on the lower link. Considering the Porter governor (Fig. 138 (a)), the forces acting on the lower link AC are: (1) the centrifugal force $\frac{W}{g}\omega^2 r$ acting radially outwards at C, (2) the vertical force W at C, (3) the tension in the upper link CB acting at C, (4) a vertical force $\frac{1}{2}W_1$ at A, (5) a horizontal force at A equal to the horizontal component of the tension in AC. Of these forces (3) and (5) are unknown, and if moments be taken about the intersection of these unknown forces, the moments of these unknown forces become zero and the moments of the other known forces must balance.

The lower link AC is reproduced in Fig. 140 and the point of intersection of the upper link CB and the horizontal through A is at O.

Let x = perpendicular distance of force $\frac{W}{g}\omega^2 r$ from O

y = ,, ,, ,, W ,, O

z = ,, ,, ,, $\frac{1}{2}W_1$,, O.

Taking moments about O and equating clockwise to counter-clockwise moments

$$\frac{W}{g}\omega^2 r \times x = W \times y + \tfrac{1}{2}W_1 \times z$$

or

$$\frac{W}{g}\omega^2 r \times \frac{x}{y} = W + \tfrac{1}{2}W_1 \times \frac{z}{y}.$$

Comparing the triangles abc (Fig. 138 (b)) and OAC (Fig. 140), it will be seen that these triangles are similar since ba is perpendicular to OA, ca is perpendicular to CA, and cb perpendicular to CO.

Hence $$\frac{x}{y} = \frac{h}{r} \quad \text{and} \quad \frac{z}{y} = \frac{ba}{bc_1} = q.$$

Substituting these values

$$\frac{W}{g}\omega^2 r \cdot \frac{h}{r} = W + \tfrac{1}{2}W_1 q,$$

or

$$\omega^2 = \frac{W + \tfrac{1}{2}W_1 q}{W} \cdot \frac{g}{h}.$$

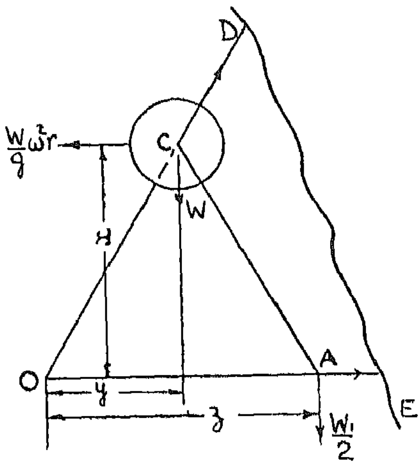


Fig. 140.

Taking friction at the sleeve into account the load at A is $\tfrac{1}{2}W_1 \pm \tfrac{1}{2}f$ depending whether the sleeve tends to move up or down, hence

$$\omega^2 = \frac{W + \tfrac{1}{2}W_1 q \pm \tfrac{1}{2}fq}{W} \cdot \frac{g}{h}.$$

This method may be applied equally well to the Proell governor (Fig. 139 (*a*)), and considering the forces acting on the lower link DCA. The unknown forces are the tension in the upper link CB and the horizontal force at A. By taking moments about the intersection of CB and the horizontal through A, moments of these unknown forces become zero, and the moments of the other forces about this point of intersection must balance.

This method is similar to the well-known method of sections used in structures for finding the load in a given member. In

Fig. 140, DE may be regarded as a cutting plane, cutting the unknown forces acting at C and A and moments about the intersection of these unknown forces become zero; the moments of the other known forces must balance.

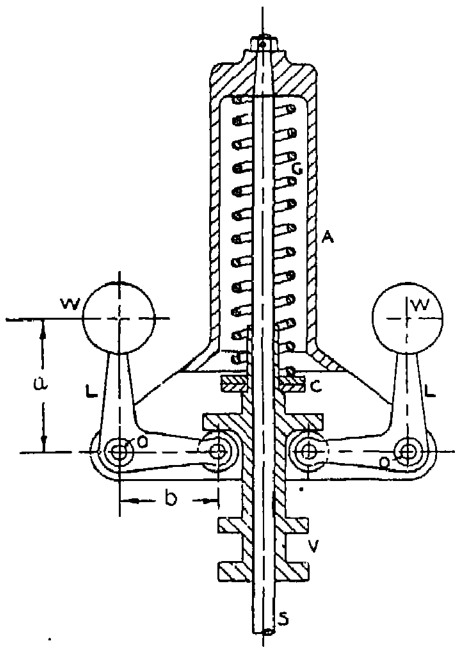


Fig. 141.

§ **178. Hartnell Governor.** A compressed spring provides a ready means of causing a load on the sleeve and a common form of governor in which a compressed spring is used is the Hartnell governor, shown in Fig. 141. The sleeve V can move up and down on the vertical spindle and the movement is limited by stops (not shown). Fixed to the spindle is a casing A, inside which a compressed spring G presses against the top of the casing and on adjustable collars C, the latter being fixed to the sleeve. The lower end of the casing is extended to carry the bell crank levers L. Each vertical arm carries a mass W and the horizontal arm a roller which can

press on the underside of a collar on the sleeve. The axis of the bell crank lever is at O.

Let W = weight of mass at end of bell crank lever,

a = length of vertical arm,

b = length of horizontal arm,

r_1 = radius to centre of mass when sleeve is in top position,

r_2 = radius to centre of mass when sleeve is in bottom position,

ω_1 = speed of governor when sleeve is in top position,

ω_2 = speed of governor when sleeve is in bottom position,

P_1 = force exerted by compressed spring at speed ω_1,

P_2 = force exerted by compressed spring at speed ω_2.

Taking moments about O and neglecting the weight of the arms of the bell crank lever and the moment of the weight of the ball,

$$\tfrac{1}{2}P_1 \times b = \frac{W}{g}\omega_1^2 r_1 \times a.$$

$$\tfrac{1}{2}P_2 \times b = \frac{W}{g}\omega_2^2 r_2 \times a.$$

These are two simultaneous equations from which P_1 and P_2 can be found for given values of a, b, r, ω_1, and ω_2.

The movement of the sleeve is directly proportional to the horizontal movement of the mass W, and in magnitude is

$$\frac{b}{a}(r_1 - r_2).$$

The difference in the forces exerted by the compressed spring in the two positions is $P_1 - P_2$, therefore the force per inch of compression is

$$\frac{P_1 - P_2}{\frac{b}{a}(r_1 - r_2)}.$$

This is known as the stiffness of the spring.

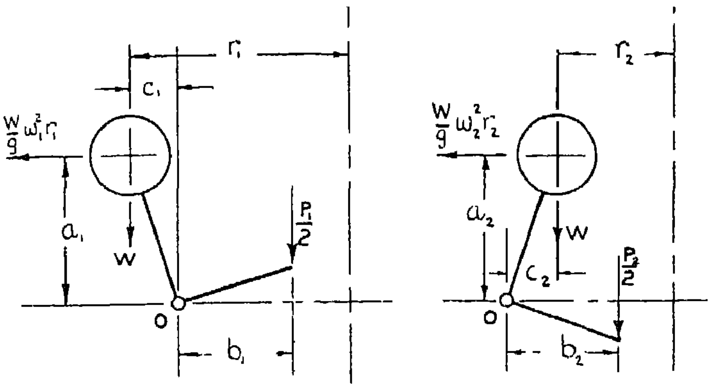
The effect of the moment of the ball may be taken into account as shown in Fig. 142 (a) and (b), in which the bell

crank lever is shown for the top and bottom positions of the sleeve respectively.

Taking moments about O,

$$\tfrac{1}{2}P_1 \times b_1 = \frac{W}{g}\omega_1^2 r_1 \times a_1 + W \times c_1.$$

$$\tfrac{1}{2}P_2 \times b_2 = \frac{W}{g}\omega_2^2 r_2 \times a_2 - W \times c_2.$$



(a) Fig. 142. (b)

Again these are two simultaneous equations for finding P_1 and P_2. The stiffness of the spring is found in the manner indicated above.

Example 4. In a spring governor of the Hartnell type the weight of each ball is 12 lb. and the lift of the sleeve is $2\frac{1}{2}$ in. The speed at which the governor begins to float is 250 revs. per min., and at this speed the radius of the ball path is $5\frac{1}{2}$ in. The mean working speed of the governor is 18 times the range of speed when the effects of friction are neglected. Calculate the stiffness of the spring, i.e. the load per inch compression. The governor is similar to Fig. 141, and $a = 6$, $b = 5$, and the centre O is 7 in. from the axis of the vertical spindle. [*Lond. B.Sc.*]

(1) Approximate solution.

Let ω = mean speed,

$\omega + \delta$ = maximum speed = ω_1,

$\omega - \delta$ = minimum speed = ω_2.

Range of speed $= (\omega+\delta)-(\omega-\delta) = 2\delta$,

$$\therefore\ \omega = 18\times 2\delta = 36\delta,$$
$$\omega_1 = \omega+\delta = 37\delta,$$
$$\omega_2 = \omega-\delta = 35\delta = \tfrac{250}{60}\times 2\pi;$$
$$\therefore\ \omega_1 = \tfrac{37}{35}\omega_2 = \tfrac{37}{35}\times\tfrac{250}{60}\times 2\pi.$$
$$\frac{b}{a}(r_2-r_1) = \text{lift of sleeve} = 2\tfrac{1}{2}\text{ in.},$$
$$\therefore\ r_1-r_2 = 2\tfrac{1}{2}\times\tfrac{6}{5} = 3\text{ in.},$$
$$r_1 = 3+5\tfrac{1}{2} = 8\tfrac{1}{2}\text{ in.}$$

Taking moments about O (Fig. 141),

$$\tfrac{1}{2}P_1\times\frac{5}{12} = \frac{12}{g}\times\left(\frac{37}{35}\cdot\frac{250}{60}\cdot 2\pi\right)^2\times\frac{8\frac{1}{2}}{12}\times\frac{6}{12},$$
$$\therefore\ P_1 = 486\text{ lb.};$$

also
$$\tfrac{1}{2}P_2\times\frac{5}{12} = \frac{12}{g}\times\left(\frac{250}{60}\times 2\pi\right)^2\times\frac{5\frac{1}{2}}{12}\times\frac{6}{12},$$
$$\therefore\ P_2 = 281\text{ lb.}$$
$$\text{Stiffness} = \frac{486-281}{2\frac{1}{2}} = \frac{205}{2\frac{1}{2}} = 82\text{ lb. per in.}$$

(2) More exact solution.

Referring to Fig. 142 (a) and (b),

$$a_1 = a_2 = 5{\cdot}81\text{ in.},$$
$$b_1 = b_2 = 4{\cdot}84\text{ in.},$$
$$c_1 = c_2 = 1{\cdot}5\text{ in.}$$

Taking moments about O,

$$\tfrac{1}{2}P_1\times\frac{4{\cdot}84}{12} = \frac{12}{g}\times\left(\frac{37}{35}\cdot\frac{250}{60}\cdot 2\pi\right)^2\times\frac{8\frac{1}{2}}{12}\times\frac{5{\cdot}81}{12}+\frac{12\times 1{\cdot}5}{12},$$
$$\therefore\ P_1 = 493\text{ lb.};$$

also
$$\tfrac{1}{2}P_2\times\frac{4{\cdot}84}{12} = \frac{12}{g}\times\left(\frac{250}{60}\cdot 2\pi\right)^2\times\frac{5{\cdot}5}{12}\times\frac{5{\cdot}81}{12}-\frac{12\times 1{\cdot}5}{12},$$
$$\therefore\ P_2 = 274\text{ lb.}$$
$$\text{Stiffness} = \frac{493-274}{2{\cdot}5} = \frac{219}{2{\cdot}5} = 87{\cdot}6\text{ lb. per inch.}$$

§ 179. Effort of a Governor. A governor running at constant speed is in equilibrium and the resultant force acting on the sleeve is zero. If the speed of the governor increases, there is a force exerted on the sleeve which tends to lift the

sleeve. The mean force acting on the sleeve for a given change of speed is the effort of the governor. The usual change of speed is 1 per cent., and since the force acting at the sleeve changes gradually from zero, when the governor is in its position of equilibrium, to a value, say E, for a 1 per cent. increase of speed, the mean force or effort is $\frac{E}{2}$.

For a Porter governor, Fig. 138, with equal arms and equal inclinations with the axis,

$$h = \frac{W+W_1}{W} \cdot \frac{g}{\omega^2}.$$

Let E = the force acting at the sleeve when the speed has increased 1 per cent., the governor height remaining the same,

$$h = \frac{W+W_1+E}{W} \cdot \frac{g}{(1 \cdot 01\omega)^2}.$$

$$\therefore \frac{W+W_1+E}{W+W_1} = \frac{(1 \cdot 01\omega)^2}{\omega^2} = 1 \cdot 02.$$

$$\frac{E}{W+W_1} = 0 \cdot 02.$$

$$\therefore E = 0 \cdot 02(W+W_1).$$

The effort, therefore, is $\frac{E+0}{2} = 0 \cdot 01(W+W_1)$.

For the general case of the Porter governor

$$h = \frac{W+\frac{W_1}{2}q}{W} \cdot \frac{g}{\omega^2}.$$

Also for a 1 per cent. increase in speed,

$$h = \frac{W+\frac{W_1}{2}q+\frac{E}{2}q}{W} \cdot \frac{g}{(1 \cdot 01\omega)^2}.$$

$$\therefore \frac{W+\frac{W_1}{2}q+\frac{E}{2}q}{W+\frac{W_1}{2}q} = \frac{(1 \cdot 01\omega)^2}{\omega^2} = 1 \cdot 02,$$

$$\frac{E}{2}q = 0{\cdot}02\left(W + \frac{W_1}{2}q\right),$$

$$E = \frac{0{\cdot}04\left(W + \frac{W_1}{2}q\right)}{q}.$$

$$\text{Effort of governor} = \frac{E+0}{2} = \frac{0{\cdot}02}{q}\left(W + \frac{W_1}{2}q\right).$$

For the Hartnell governor when in a position of equilibrium and neglecting the moment of the weight of the rotating mass

$$\frac{P}{2} \times b = \frac{W}{g}\omega^2 r \times a.$$

Let $E =$ the force acting at the sleeve when the speed has increased 1 per cent. before the configuration of the governor alters.

$$\left(\frac{P}{2} + \frac{E}{2}\right)b = \frac{W}{g}(1{\cdot}01\omega)^2 r \times a.$$

$$\therefore \frac{P+E}{P} = \frac{(1{\cdot}01\omega)^2}{\omega^2} = 1{\cdot}02,$$

$$E = 0{\cdot}02P.$$

The effort, therefore, is $\frac{E+0}{2} = 0{\cdot}01P$.

§ **180. Sensitiveness of a Governor.** A governor is said to be sensitive when it readily responds to a small alteration of speed, or, for purposes of comparison, when the change of speed is small for a given displacement of the sleeve. A governor only operates between certain speeds corresponding to the positions of the sleeve when up against the top stop and when resting on the bottom stop. Above and below these speeds the governor does not function. The difference between these speeds is the range of speed and the sensitiveness is sometimes measured by the ratio of the mean speed of the governor and the range of speed.

$$\text{Sensitiveness} = \frac{\text{mean speed}}{\text{range of speed}}.$$

While sensitiveness is a desirable quality in certain types of

engines, a governor that is too sensitive will not necessarily be effective in keeping the speed of the engine more or less constant. If a governor is too sensitive a slight alteration of load upon the engine will cause a large displacement of the sleeve with a consequent alteration of speed, and again the governor operates causing another alteration of speed. These fluctuations of speed are known as *hunting*.

A governor in which the range of speed is zero is termed an *isochronous* governor. Since the range of speed is zero, an isochronous governor is infinitely sensitive and the governor has the same speed for all positions of the sleeve. In practice an isochronous governor is an impossibility on account of the friction at the sleeve.

Neglecting friction, for a Porter governor running at a speed ω_1,

$$\omega_1^2 = \frac{W+q\dfrac{W_1}{2}}{W}\cdot\frac{g}{h_1};$$

and at speed ω_2

$$\omega_2^2 = \frac{W+q\dfrac{W_1}{2}}{W}\cdot\frac{g}{h_2}.$$

For the condition of isochronism $\omega_1 = \omega_2$, and therefore $h_1 = h_2$; but this is impossible for the configuration of a Porter governor, hence a Porter governor cannot possibly be isochronous.

For a Hartnell governor, at speed ω_1,

$$\frac{P_1}{2}\times b = \frac{W}{g}\omega_1^2 r_1\times a;$$

and for speed ω_2,

$$\frac{P_2}{2}\times b = \frac{W}{g}\omega_2^2 r_2\times a.$$

For a condition of isochronism, $\omega_1 = \omega_2$.

$$\therefore \frac{P_1}{} = \frac{r_1}{}.$$

EXAMPLE 5. In a spring-controlled governor the radial force acting on the balls was 900 lb. when the centre of the balls was 8 in. from the axis, and 1,500 lb. when at 12 in. Assuming that the force varies directly as the radius, find the radius of the ball

path when the governor runs at 270 revs. per min. Also find what alteration in the spring load is required in order to make the governor isochronous, and the speed at which it would then run. Weight of each ball = 60 lb. [*Lond. B.Sc.*]

Let the law connecting radial force and radius be of the form

$$F = ar+b,$$

$$900 = \tfrac{8}{12}a+b,$$

$$1{,}500 = \tfrac{12}{12}a+b.$$

$$\therefore\ 600 = \tfrac{4}{12}a,$$

$$a = 1{,}800,$$

$$\text{whence } b = -300.$$

$$\therefore\ F = 1{,}800r-300 = \frac{W}{g}\omega^2 r,$$

$$1{,}800r-300 = \frac{60}{g}\cdot\left(\frac{270}{60}\times 2\pi\right)^2 r,$$

$$\text{whence } r = 0{\cdot}967 \text{ ft.} = 11{\cdot}6 \text{ in.}$$

For the governor to be isochronous F must vary directly as r, since $\dfrac{P_1}{P_2}=\dfrac{r_1}{r_2}=\dfrac{8}{12}$. This is accomplished by making $b = 0$ in the relation $F = ar+b$. Thus the load in the spring must be increased such that the radial force is increased by 300 lb., i.e the load in the spring at 8 in. radius must be increased by $\frac{1}{3}$ of the original load.

For the governor to be isochronous $F = 1{,}200$ lb. at 8 in. radius; hence

$$\frac{60}{g}\times\frac{8}{12}\omega^2 = 1{,}200,$$

and

$$\omega = 31{\cdot}1 \text{ radians per sec.}$$

$$N = 297 \text{ revs. per min.}$$

§ **181. Over-compression of Spring.** The initial compression of the spring affects not only the mean speed at which a Hartnell governor will run but also affects the range of speed over which the governor will operate. As the initial compression of the spring increases, the mean speed of the governor also increases but the range of speed decreases until with a given initial compression the governor becomes isochronous and runs at the same speed for all positions of the sleeve. In this case the range of speed becomes zero. Further

initial compression of the spring causes the governor to run at a higher speed with the sleeve in the lower position than the speed corresponding to the higher position of the sleeve.

To illustrate these points still further, consider a Hartnell governor similar to that shown in Fig. 141.

Let $a = 6$ in.; $b = 4$ in.; $W = 12$ lb.;

radius of $O = 5$ in.; lift of sleeve $= 2$ in.

Stiffness of spring $= 80$ lb. per inch of compression.

The arms of the bell-crank lever are assumed vertical and horizontal when the sleeve is in its mid position.

Alteration of radius $= \frac{6}{4} \times 2 = 3$ in.

Hence $r_1 = 5 + \frac{3}{2} = 6\frac{1}{2}$ in.; $r_2 = 5 - \frac{3}{2} = 3\frac{1}{2}$ in.

The extreme positions of the arms are as shown in Fig. 142.

Using the same notation as in § 178 and neglecting the small correction for the moment of the weight of the ball and also neglecting friction, the following equations are found by taking moments about O:

$$\frac{W}{g}\omega_1^2 r_1 \times a = \frac{P_1}{2} \times b,$$

$$\frac{W}{g}\omega_2^2 r_2 \times a = \frac{P_2}{2} \times b.$$

Substituting values

$$\frac{12}{32{\cdot}2} \times \omega_1^2 \times \frac{6\frac{1}{2}}{12} \times \frac{6}{12} = \frac{P_1}{2} \times \frac{4}{12},$$

and
$$\frac{12}{32{\cdot}2} \times \omega_2^2 \times \frac{3\frac{1}{2}}{12} \times \frac{6}{12} = \frac{P_2}{2} \times \frac{4}{12}.$$

Hence $\omega_1^2 = 1{\cdot}65 P_1$

and $\omega_2^2 = 3{\cdot}07 P_2$.

For isochronism $\dfrac{P_1}{P_2} = \dfrac{r_1}{r_2}$.

Hence $\dfrac{(x+2)80}{x80} = \dfrac{6\frac{1}{2}}{3\frac{1}{2}}$, where $x =$ initial compression, from which $x = 2{\cdot}33$ in.

Values of P_1 and P_2 depend upon the initial compression of the spring and by taking a series of values of the initial

compression corresponding values of ω_1^2 and ω_2^2 can be found. The results can be conveniently summarized in the form of a table as follows:

Initial Compression in.	P_1 *lb.*	P_2 *lb.*	$\omega_1^2 = 1{\cdot}65P_1$	$\omega_2^2 = 3{\cdot}07P_2$	N_1	N_2	*Range of speed* $N_1 - N_2$
1	240	80	396	245	190	149	41
$1\frac{1}{2}$	280	120	462	368	206	183·5	22·5
2	320	160	529	491	220	212	8
2·33	346	186	571	571	228·5	228·5	0
$2\frac{1}{2}$	360	200	594	614	233	237	−4
3	400	240	660	737	246	260	−14

In the above table values of P_1 and P_2 are found from the initial compression and the lift of the sleeve; when the initial compression is 2 in. $P_2 = 2 \times 80 = 160$ lb. and since the lift of the sleeve is 2 in. the compression is 4 in. when the sleeve is in the top position and $P_1 = 4 \times 80 = 320$ lb. Columns 6 and 7 are found from values of ω_1^2 and ω_2^2, thus $N_1 = \omega_1 \times \frac{60}{2\pi}$. In the last column the range of speed over which the governor operates is shown as the difference between N_1 and N_2.

Considering the results in the last column it will be seen that the range of speed decreases with the increase of initial compression culminating in a condition of isochronism when the initial compression is 2·33 in. Further initial compression gives a greater speed for the bottom position of the sleeve and the range of speed becomes negative; in this condition the governor is said to be unstable. Values of N_1 and N_2 are shown plotted against the initial compression in Fig. 143. The vertical distance between the two curves gives the range of speed over which the governor operates for any given initial compression, and the point of intersection gives the speed and initial compression for isochronism.

The curve *ab* shows the relation between values of N_1, the speed corresponding to the top position of the sleeve, and the initial compression; the curve *cd* shows the relation between N_2, the speed corresponding to the bottom position of the sleeve, and the initial compression. The vertical distance *ef* shows the range of speed when the initial compression is

$1\frac{1}{2}$ in. and the point of intersection of the two curves at g gives the initial compression when the range of speed is zero, i.e. when the governor is isochronous.

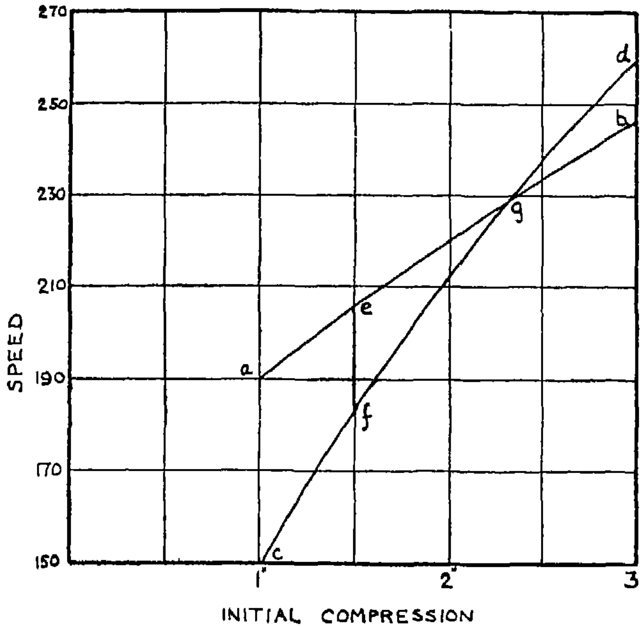


Fig. 143.

§ **182. Controlling Force.** The force acting radially inwards upon each rotating mass is known as the controlling force. The controlling force is supplied by the weight of the rotating mass, weight of sleeve, and central load in the case of Watt and Porter governors, and by the compressed spring in the case of governors of the Hartnell type.

By constructing a diagram on which lines are drawn representing the centrifugal force for all radii of the revolving ball at various speeds and superimposing on this diagram a line representing controlling force for all radii, more complete information can be obtained than is possible with a mathematical analysis for one or two positions of the sleeve. Further, by adding controlling force lines representing friction

at the sleeve, the effect of this friction can be readily seen at all speeds and radii. Since the controlling force is the force acting radially inwards on each rotating ball it becomes exactly equal in magnitude but opposite in direction to the centrifugal force on each ball. The controlling force depends upon the weight of the rotating ball, the central load, and the configuration of the governor. The centrifugal force depends upon the rotating weight, the speed, and configuration. Diagrams constructed in the above manner are known as controlling force diagrams.

§ 183. Controlling Force Diagram for Porter Governor. For the Porter governor shown in Fig. 138 (*a*), and considering the case when the arms are equal and equally inclined to the axis of the vertical spindle,

$$\omega^2 = \frac{W+W_1}{W} \cdot \frac{g}{h}.$$

Hence
$$\frac{W}{g}\omega^2 r = \frac{W}{g} \cdot r \cdot \frac{W+W_1}{W} \cdot \frac{g}{h}$$
$$= (W+W_1)\frac{r}{h}.$$

The expression $\frac{W}{g}\omega^2 r$ is the centrifugal force acting on each rotating ball and consequently $(W+W_1)\frac{r}{h}$ represents the controlling force. If the effect of friction is considered it has been seen that W_1 is modified to $W_1 \pm f$, where f represents the friction at the sleeve. Thus

$$\text{controlling force} = (W+W_1 \pm f)\frac{r}{h}.$$

To illustrate the construction of a controlling force diagram consider the Porter governor in Examples 2 and 3 at the end of § 175.

Lines representing the centrifugal force on each ball are first drawn for various speeds and radii:

$$\text{centrifugal force} = \frac{W}{g}\omega^2 r.$$

Since the centrifugal force depends upon the radius, for a given speed the line will be a straight line between the limits of radii. Thus at 6 in. radius

$$\text{centrifugal force} = \frac{6}{32 \cdot 2} \cdot \omega^2 \times \frac{6}{12} = 0 \cdot 0932\omega^2$$

at 8 in. radius centrifugal force $= \dfrac{6}{32 \cdot 2} \cdot \omega^2 \times \dfrac{8}{12} = 0 \cdot 1242\omega^2$.

Taking a series of values of ω corresponding to speeds varying from 150 to 190 revolutions per minute, values of centrifugal force are readily calculated and for convenience are shown in tabular form, thus

N *revs. per min.*	ω	ω^2	*Cent. Force* $r = 6$ *in.*	*Cent. Force* $r = 8$ *in.*
150	15·7	246	22·9	30·6
155	16·22	264	24·6	32·8
160	16·75	281	26·2	34·9
165	17·28	299	27·9	37·2
170	17·8	317	29·5	39·4
175	18·32	336	31·3	41·7
180	18·85	355	33·1	44·1
185	19·37	375	34·9	46·6
190	19·90	396	36·9	49·1

These results are shown plotted against radius in Fig. 144 by straight diverging lines.

$$\text{Controlling force} = (W + W_1 \pm f)\frac{r}{h}.$$

Taking values of r from 6 in. to 8 in. and calculating the value of h, the controlling force is readily tabulated thus:

r *in.*	$h = \sqrt{(100 - r_2)}$ *in.*	*Controlling force*		
		Neglecting friction	*Taking friction into account*	
		$= (W + W_1)\frac{r}{h}$	$(W + W_1 + f)\frac{r}{h}$	$(W + W_1 - f)\frac{r}{h}$
6	8	25·5	27·8	23·2
6·5	7·6	29·1	31·7	26·5
7	7·14	33·4	36·3	30·4
7·5	6·61	38·6	42·0	35·2
8	6	45·4	49·4	41·4

These results are also shown plotted against radius in Fig. 144 in which *acb* represents the controlling force neglecting friction. When the controlling force has the value at *a*, the corresponding centrifugal force is at a speed of about 158 revolutions per minute; at *c* where the radius is 7·2 in.

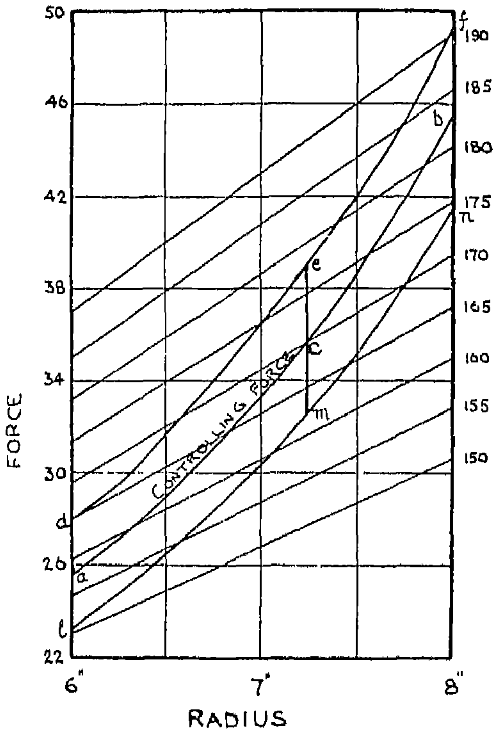


Fig. 144.

the controlling force line cuts the centrifugal force line corresponding to a speed of 170 revolutions per minute; at *b* the speed is 182·5 revolutions per minute.

Taking friction at the sleeve into account, when the sleeve tends to move up, friction acts downwards and is thus equivalent to an additional central load. The line *def* represents controlling force when the sleeve tends to move up; similarly *lmn* represents the controlling force when the sleeve

tends to move down. The range of speed is now represented by the points f and l corresponding to speeds of 191 and 151 revolutions per minute.

The vertical intercept em represents the change of controlling force for a radius of 7·2 in.; the corresponding centrifugal forces are at speeds of 178 and 162 revolutions per minute and the range of speed over which the governor does not alter its configuration is thus 16 revolutions per minute.

§ 184. Controlling Force Diagram for Hartnell Governor. The controlling force for a spring-controlled Hartnell governor is supplied by the compression of the central spring. Referring to the governor shown in Fig. 141

$$\frac{W}{g}\omega^2 r \times a = \frac{P}{2} \times b.$$

Hence
$$\frac{W}{g}\omega^2 r = \frac{P}{2} \times \frac{b}{a}.$$

Thus $\frac{P}{2} \times \frac{b}{a}$ is the controlling force and the diagram is constructed in a manner similar to that described in § 183. Fig. 145 is constructed from data to the governor in § 181. Values of centrifugal force are shown in tabular form thus:

Speed	ω	ω^2	Centrifugal force	
			$r_1 = 6\frac{1}{2}$ *in.*	$r_2 = 3\frac{1}{2}$ *in.*
140	14 66	215	43·4	23·4
160	16·75	281	56·7	30·5
180	18·85	355	71·7	38·6
200	20 95	440	88·9	47·8
220	23 05	531	107·2	57·7
240	25·1	630	127·2	68·5
260	27·2	740	149·4	80·4

These results are shown plotted in Fig. 145 by thin diverging lines.

Controlling force lines for different initial compressions of the spring can now be found and plotted on the same diagram.

$$\text{Controlling force} = \frac{P}{2} \times \frac{b}{a} = \frac{P}{2} \times \frac{4}{6} = \frac{P}{3}.$$

When the initial compression is 1 in., the controlling force for

bottom position of the sleeve is $\frac{80}{3}$ or 26·7 lb. For top position, when the left of the sleeve is 2 in. and the compression is therefore 3 in., the controlling force is $\frac{80 \times 3}{3}$ or 80 lb. Taking

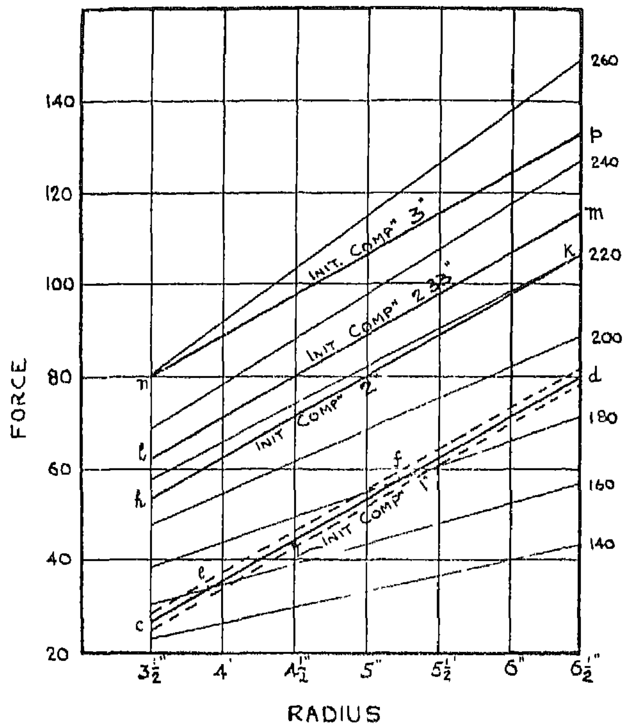


Fig. 145.

additional values of the initial compression of 2, 2·33, and 3 in. the tabulated results are as follows:

Initial compression in.	Controlling force $3\frac{1}{2}$ in.	$r_1 = 6\frac{1}{2}$ in.
1	26·7	80
2	53·3	106·7
2·33	62·1	115·6
3	80	133

These values of controlling forces are shown in thick lines

in Fig. 145. Considering the case when the initial compression is 1 inch, the controlling force line cuts lines of centrifugal force at points corresponding to speeds 160 and 180 revolutions per minute in the points *e* and *f*. The point *e* signifies equilibrium between the controlling force and centrifugal force when the speed is 160 revolutions per minute and the radius of the ball is almost 4 inches. Similarly, the ball radius for a speed of 180 revolutions per minute is at *f* representing 5·2 inches radius. The points *c* and *d* represent the range of speed of the governor when the effect of friction at the sleeve is neglected, and the limiting speeds are 190 and 149 revolutions per minute giving a range of speed of 41 revolutions per minute.

The line of controlling force for an initial compression of 2 in. is represented by *hk*, and it will be readily seen that the slope of this line nearly coincides with the slope of the centrifugal force line, indicating that the range of speed is less and that the governor is nearing the point of isochronism. When the initial compression is 2·33 in., the controlling force line is *lm*, and the slope coincides with the centrifugal force line for a speed of about 228·5 revolutions per minute. For an initial compression of 3 in. the line *np* has a slope less than that of the corresponding centrifugal force and this is a clear indication of initial over-compression of the spring. From the diagram it is thus seen that when the line of controlling force cuts lines of centrifugal force at points corresponding to higher speeds with increase of radius, the governor is stable; when the line of controlling force is parallel to the centrifugal force line a condition of isochronism is implied, and when the line of controlling force cuts a line of centrifugal force at a lower speed for an increase of radius, the spring is initially compressed too much for stable equilibrium.

The effect of friction at the sleeve can also be shown by drawing additional lines of controlling force to take into account this friction. Two additional lines are shown dotted for an initial compression of 1 inch and for friction equivalent to a force of 5 lb. at the sleeve. The speed corresponding to a radius of $6\frac{1}{2}$ in. is increased to 192 and the speed corresponding

to a radius of $3\frac{1}{2}$ in. is decreased to 145, giving a range of speed of 47 revolutions per minute.

Neglecting the effect of friction a governor has a position of equilibrium for any given speed and this position determines the position of the sleeve. Taking friction into account, for any given position of the sleeve there are two extreme speeds at which the governor will run and over this range of speed the position of the sleeve is not altered. This applies to both top and bottom positions of the sleeve and consequently, taking the upper limit when the sleeve is in its top position and the lower limit when the sleeve is in its bottom position, the effect is to increase the range of speed over which the governor can operate.

EXERCISES. XI

1. In a Porter governor the arms are equal and 15 in. long; the points of suspension are at equal distances of 3 in. from the axis of the governor. Each of the rotating balls weighs 4 lb. Determine the central load in order that the speed may be 200 revs. per min. when the arms are at right angles.

If the friction at the sleeve is equivalent to a force of 10 lb. acting at the sleeve, find the speed at which the governor will begin to move (*a*) up, (*b*) down.

2. In a Proell governor similar to Fig. 139 (*a*), $AC = BC = 8$ in.; $CD = 3$ in., and the points A and B are $1\frac{1}{2}$ in. from the axis of the governor. When the sleeve is on the bottom stop, the vertical distance $AB = 13$ in. and CD is vertical. Each ball weighs 6 lb. Determine the central load W so that the sleeve just leaves the bottom stop at 120 revs. per min., and find the speed when the sleeve is $1\frac{1}{2}$in. above the bottom stop.

3. In Question 2, if the arm CD is eliminated and the ball is at C, the governor becomes a Porter governor. Solve for the same quantities as in Question 2.

4. A Watt governor has crossed arms 10 in. long, and each arm is pivoted $1\frac{1}{4}$ in. from the axis of revolution. Neglecting the weight of the arms and assuming each ball to weigh 10 lb., draw the controlling force diagram of the governor for radii between 5 in. and 7 in. [*Inst. C. E.*]

5. Obtain an expression for the height of a simple governor of the Watt type in terms of the speed of revolution in radians per second.

What are the effects of friction and of adding a central weight to the sleeve of a Watt governor?

6. Find the maximum and minimum speeds for a crossed arm governor. The apex angle is 60° at its lowest and 120° at its highest position. Length of arms 10 in. The suspension pins are 1·5 in. from the axis of revolution. [*Inst. C. E.*]

7. A Watt type of governor has balls weighing 29 lb. each, their centres being 14 in. from the suspension pin. Two helical springs connect the ends of two pins which pass through the centres of the balls and are parallel to the suspension pin. The unstretched length of the springs is 10 in., and when the balls are in their lowest position the springs are 14 in. long and 16 in. when the balls are right out. The spring stretches 1·5 in. per 100 lb. Find the speed when the governor begins to lift and when at its extreme outer position. [*Inst. C. E.*]

8. In a spring governor of the Hartnell type (Fig. 141) the lengths of the arms a and b are 6 in. and 5 in. respectively. When in mid position the centres of the balls are at 7 in. radius, and the speed of the governor is 300 revs. per min. Each ball weighs 12 lb., and the force necessary to compress the spring is 120 lb. per in. Determine the speeds of the governor in revolutions per minute when the ball radius is 8 in. and 6 in. respectively. The weight and obliquity of the arms may be neglected. [*Inst. C. E.*]

9. In a spring-controlled governor of the Hartnell type (Fig. 141) the radius of revolution in mid position is 7 in., $a = 6$ in., $b = 5$ in. Each ball weighs 12 lb., and the total lift of the sleeve is $2\frac{1}{2}$ in. If the stiffness of the spring is 88 lb. per in., find the speeds for the top and bottom stops when the initial compression is (a) 4·1 in.; (b) 4·6 in.; (c) 4·92 in.

10. Calculate the stiffness and compression of a spring for the Hartnell governor in Question 9, with which a variation of 1 per cent. in the speed will cause a change of radius from $6\frac{1}{2}$ to $7\frac{1}{2}$ in., the mean speed being 300 revs. per min.

11. A Hartnell governor (Fig. 141) is required to revolve at the speeds 297 and 303 revs. per min. when the radius of revolution of the centres of the balls is 5 in. and 8 in. respectively; the radius of the balls in mid position is $6\frac{1}{2}$ in. and $a = 5\frac{1}{2}$ in., $b = 4\frac{1}{2}$ in. Find the mean change in the force on the spring per inch of its compression between the given limits, allowing for the effect of gravity on the balls. Also allowing 2 per cent. extra for frictional resistance, find the speeds when (a) the sleeve is rising, (b) the sleeve is falling. The weight of each ball is 9 lb. [*Lond. B.Sc.*]

12. What is the object of loading a governor? In a governor of the Porter type, the governor balls each weigh 4 lb., and the loading carried by the sleeve is 40 lb. The links form an equal-sided parallelogram in all positions, and each link is 10 in. long. At what speed must the governor be driven so that the balls revolve in a circle of 8 in. radius? [*Lond. B.Sc.*]

13. A loaded governor of the Porter type has equal arms and links each 12 in. long. The rotating balls each weigh 4 lb. and the central weight is 60 lb. The drag on the sleeve due to friction, valve resistance, etc., is 3 lb. If the limits to the movement of the arms are 45° and 30° respectively to the vertical, calculate the revolutions per minute between which this governor is operative.

Also find the range of speed for a half-way position of the arms, viz. $37\frac{1}{2}°$ to the vertical. [*Lond. B.Sc.*]

14. A Porter governor carries a central load of 50 lb., and each ball weighs 8 lb. The upper links are each 8 in. long and the lower links 12 in. The points of suspension of the upper and lower links are $2\frac{1}{2}$ in. from the axis of the vertical spindle. Calculate the speed of the governor in revolutions per minute if the radius of revolution of the governor balls is 5 in. At what speed would the balls begin to move outward if the frictional resistance at the sleeve is equivalent to 10 lb.? [*Lond. B.Sc.*]

15. Find the maximum and minimum speeds of a simple governor in which the maximum and minimum radii of the ball paths are 9 in. and 6 in. respectively, and in which the arms, 24 in. long, are pivoted at points 2 in. from the axis of rotation: (*a*) for an open armed governor, (*b*) for a crossed arm governor. [*Lond. B.Sc.*]

16. A Porter governor has equal arms each 14 in. long. The points of suspension of the arms are $1\frac{1}{2}$ in. from the vertical axis, and the upper and lower arms are at 90° to each other. Neglecting friction, determine the central load for a speed of 200 revs. per min. Each ball weighs 4 lb. If the force required to overcome frictional resistances at the sleeve is 8 lb., find the speeds at which the governor will rise and fall respectively. [*Lond. B.Sc.*]

17. In a Porter governor the balls each weigh 4 lb., the central weight is 48 lb., and the links are equal in length. The governor is running at 200 revs. per min. If the frictional resistance to the movement of the sleeve in the direction of its motion is equal to a force of 4 lb., determine the maximum and minimum speeds of the governor before the sleeve begins to move up or down. [*Lond. B.Sc.*]

18. Prove that the height in inches of a loaded governor of the Porter type with equal arms is given by the formula

$$H = \frac{W+w}{w} \times \frac{9 \cdot 79}{n^2}.$$

H = height of governor in inches,
W = weight of the added load,
w = weight of one of the balls,
n = revolutions of the governor per second.

Given that $w = 3$ lb., $W = 45$ lb., and that the speed of the governor rises suddenly from 240 to 245 revs. per min., find the mean lifting force on the sleeve. [*Lond. B.Sc.*]

19. In a loaded governor of the type in which the load is supported directly by the revolving balls, see Fig. 146 (a), the arms are 12 in. long. Each ball weighs 5 lb., and the central load is 85 lb. In their lowest position the arms are inclined at 25° to the axis of the central spindle. The frictional resistance of the governor is equivalent to a vertical force of 5 lb. on the sleeve. Determine the lift of the sleeve in order that the maximum descending speed shall be equal to the minimum ascending speed. What will then be the range of speed for this governor? [*Lond. B.Sc.*]

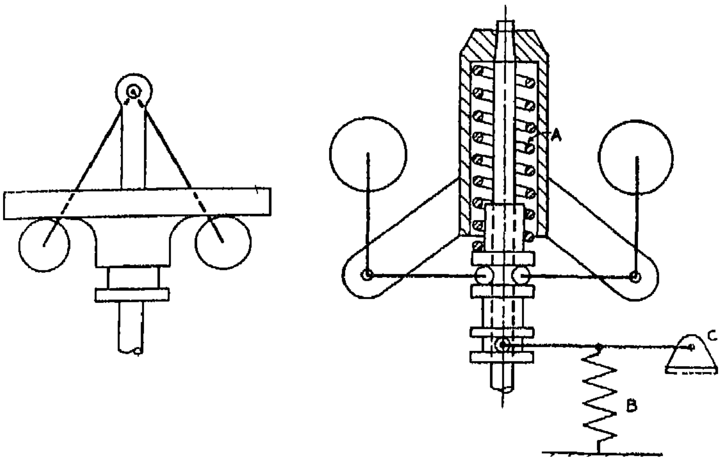


Fig. 146. (b)

20. When running at its normal speed of 180 revs. per min. the height of a Porter governor is 8 in. If the arms and links are equal, what is the ratio between the weight of the load and that of one ball?

What will be the percentage increase of speed of this governor, before the balls begin to rise, if the frictional resistance acting at the sleeve is taken as $\frac{1}{12}$ of the load? [*Lond. B.Sc.*]

21. The Hartnell type spring-loaded governor shown in Fig. 146 (b) has two balls of 10 lb. each, which revolve in a circle of 16 in. diameter when the sleeve is at mid-travel. The total movement of the sleeve is 1 in., which is the same as that of the balls. An adjustable load is applied to the sleeve by means of the spring B attached to the centre of a lever moving about a fixed centre C in the manner shown; the tension in B being adjusted by hand. When there is no tension in the spring B, the sleeve just commences to rise from its lowest position at a speed of 200 revs. per min., and reaches the upper limit of its travel at 208 revs. per min. Find the load-extension scale of the spring A, and also for the spring B, which will make these two limiting

speeds 220 and 232 revs. per min. Neglect the effect of gravity on the balls. [*Lond. B.Sc.*]

22. A Porter governor has equal links 10 in. long; each ball weighs 5 lb., and the load is 25 lb. When the ball radius is 6 in. the valve is full open, and when the radius is $7\frac{1}{2}$ in. the valve is closed. Find the maximum speed and the range of speed. If the maximum speed is to be increased 20 per cent. by an addition to the load, find what addition is required. [*Lond. B.Sc.*]

23. In a Porter governor the upper and lower arms are each 10 inches long and the points of attachment to the top collar and the sleeve are on the axis of the vertical spindle. If the central load is W_1 lb., and each of the weights attached to the arms is W lb., find the ratio of W_1 to W when the speed of the governor is 200 revolutions per minute and the arms are inclined at 30 degrees to the vertical axis.

If the friction of the governor is equivalent to a force of $\frac{W}{4}$ lb. at the sleeve, find the range of speed before the sleeve begins to move. [*Lond. B.Sc.*]

24. Derive an expression for the height of a Porter governor with a central load.

In such a governor the upper and lower arms are each 10 in. long and they are attached at points on the central spindle. In the bottom position of the sleeve the arms make an angle of 35 degrees with the vertical spindle and in the top position an angle of 45 degrees. Assuming that the speed of the governor for mid-position of the sleeve is the arithmetical mean of the speeds for top and bottom positions of the sleeve, find the ratio of the range of speed to the mean speed.

If the weight of the central load is 8 times the weight of each of the rotating masses find the mean speed of the governor. [*Lond. B.Sc.*]

25. In a spring-controlled governor of the Hartnell type the arms are at right angles. The vertical arm is $6\frac{1}{2}$ inches long and the horizontal arm $5\frac{1}{2}$ inches. The masses attached to the ends of the vertical arms weigh 7 lb. each and the radius of these masses is $6\frac{1}{2}$ inches when the sleeve is in its mid-position, the arms then being vertical and horizontal. Find the strength and initial compression of the central spring when the speed fluctuates from 180 to 200 revolutions per minute for a sleeve lift of 1 inch.

If friction is equivalent to a load of 5 lb. at the sleeve, find the total alteration in speed that may occur before the sleeve begins to move from its mid-position. [*Lond. B.Sc.*]

CHAPTER XII

BALANCING

§ 185. Forces due to a Revolving Mass. It has already been explained, § 12, that a particle or mass moving in a circular path has a centripetal or radial acceleration of $\omega^2 r$. The force required to produce this acceleration is $\frac{W}{g}\omega^2 r$ and is known as the *centripetal* force. The reaction of this force on the centre of rotation is the *centrifugal* force, which is equal

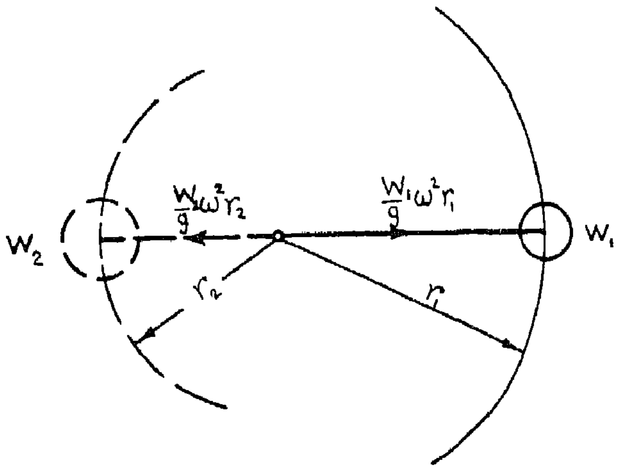
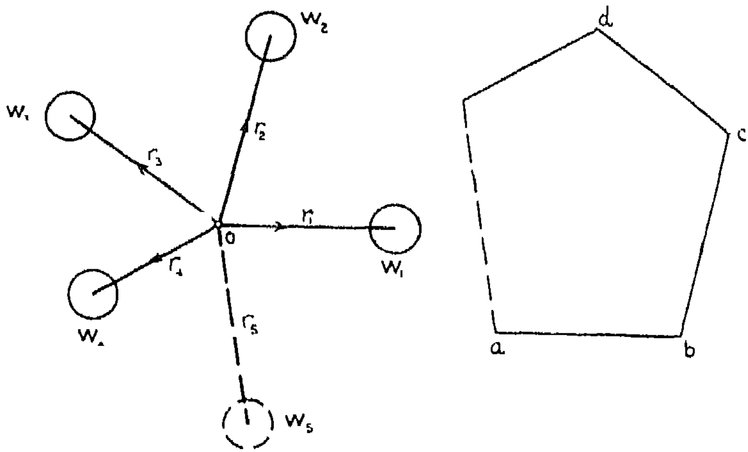


Fig. 147.

in magnitude to the centripetal force. In Fig. 147, let W_1 be a mass rotating in a circular path of radius r_1 at a constant speed of ω radians per second. The disturbing force on the centre of rotation O is $\frac{W_1}{g}\omega^2 r_1$. The magnitude of this force remains constant for all positions of W_1, but the direction alters as W_1 moves round in its circular path. The direction is always away from the centre of rotation and on a line joining the centre and the mass centre of W_1. For the centre of rotation to be in equilibrium a force equal and opposite to $\frac{W_1}{g}\omega^2 r_1$ must be applied, and this may be accomplished by

placing a mass of weight W_2 at a radius r_2, diametrically opposite to W_1 and rotating at the same speed as W_1, such that $\frac{W_2}{g}\omega^2 r_2 = \frac{W_1}{g}\omega^2 r_1$. The mass W_2 at radius r_2 is then said to balance the mass W_1 at radius r_1 and the centre of rotation is in equilibrium. Since each of the masses is rotating at constant speed ω, the term $\frac{\omega^2}{g}$ is common to both masses, and the condition for equilibrium is $W_2 r_2 = W_1 r_1$.



(a) Fig. 148. (b)

In any given system of rotating masses, each is rotating at a speed ω and the term $\frac{\omega^2}{g}$ is common to all; hence it is usual to employ the term Wr in place of $\frac{W}{g}\omega^2 r$ when finding a balancing mass; in the evolution of the disturbing force the term $\frac{\omega^2}{g}$ must be included.

§ 186. Several Masses revolving in One Plane. In Fig. 148 (*a*), let W_1, W_2, W_3, and W_4 be masses rotating about the centre O with constant angular speed. Let the radii be r_1, r_2, r_3, and r_4 respectively. The forces acting on the centre O are proportional to $W_1 r_1$, $W_2 r_2$, $W_3 r_3$, and $W_4 r_4$ respectively,

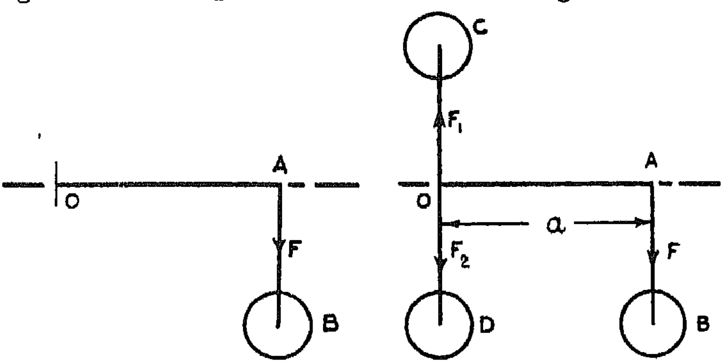
and the resultant of these forces is obtained by drawing a vector diagram as in Fig. 148 (*b*). The vector *ab* is drawn parallel to OW_1 to represent $W_1 r_1$, *bc* parallel to OW_2 to represent $W_2 r_2$, *cd* parallel to $W_3 r_3$ to represent $W_3 r_3$, and $W_4 r_4$ parallel to OW_4 to represent $W_4 r_4$. The sum of these vectors is *ae*, and hence the vector *ea* is the equilibrant and represents a force, which, acting in a direction from *e* to *a* and of magnitude proportional to *ea*, would balance the forces $W_1 r_1$, $W_2 r_2$, $W_3 r_3$, and $W_4 r_4$. A mass W_5 at radius r_5 placed in such a position that $W_5 r_5$ is represented by *ea* would balance the masses W_1, W_2, W_3, and W_4 in the positions given. The position of W_5 is shown in Fig. 148 (*a*) in which OW_5 is drawn parallel to *ea*.

Since *ea* represents the equilibrant of the four forces, the five forces represented by *ab*, *bc*, *cd*, *de*, and *ea* are in equilibrium and their sum is zero, i.e. $ab+bc+cd+de+ea = 0$.

or, vector sum of $W_1 r_1 + W_2 r_2 + W_3 r_3 + W_4 r_4 + W_5 r_5 = 0$.

In a general case, the condition for equilibrium of a number of masses rotating in one plane is that the sum of the product $W \times r$ (for each mass) shall be equal to zero. Stated mathematically, $\Sigma\, Wr = 0$.

§ **187. Reference Plane.** When several masses are rotating in different planes a further disturbing influence is



(*a*) Fig. 149. (*b*)

experienced due to the formation of couples. In Fig. 149 (*a*), let *OA* represent the axis of a shaft and *B* a mass of weight *W* attached to the shaft at a radius *r*. Let *F* be the centrifugal

force acting on the shaft when rotating at ω radians per second, F being equal to $\frac{W}{g}\omega^2 r$. If B is the only mass attached to the shaft, the tendency of the force F is to displace the shaft in the direction of the force F, which is, of course, changing in direction as the shaft rotates. The shaft may be balanced by placing a mass at A diametrically opposite to B and of magnitude such that its centrifugal force is equal to that of B. In many cases, however, it is not convenient to place the balance mass in the plane of rotation of the rotating mass B.

If the shaft is assumed to be merely supported at any point such as O, the disturbing effect of F acting at A is not affected in any way by two equal and opposite forces, F_1 and F_2, acting at O as in Fig. 149 (b). These two equal and opposite forces F_1 and F_2 can be conveniently applied by attaching masses C and D to the shaft such that their respective centrifugal forces are each equal to F. The two forces F_1 and F together form a couple of magnitude $F \times a$ or $F_1 \times a$, where a is the distance between O and A. This couple tends to rock the shaft in a plane which contains F_1 and F. The force F_2 acting on O tends to displace the shaft in the direction of F_2 or F.

A plane, perpendicular to the axis of the shaft passing through a point such as O, is called a *reference* plane, and by applying forces F_1 and F_2 at O, the mass B is said to be transferred to the reference plane. The effect of transferring a rotating mass in one plane to a reference plane is to cause a force of magnitude equal to the centrifugal force of the rotating mass to act in the reference plane together with a couple of magnitude $F \times a$, where a is the distance between the plane of rotation and the reference plane.

Similarly, other masses acting in different planes can be transferred to the reference plane, thereby inducing forces in the reference plane and couples about the reference plane. For complete equilibrium of a number of rotating masses in different planes the forces in the reference plane must balance and the couples about the reference plane must balance.

§ **188. Representation of a Couple by a Vector.** In Fig. 150, let OA represent a shaft, and let a plane through O

perpendicular to the shaft be the reference plane. A mass B is rotating in a plane through A and perpendicular to the shaft. It has been seen that two equal and opposite forces F_1 and F_2, due to rotation of masses C and D, may be applied at O without altering the force F already acting on the shaft due to the rotation of B. The forces F and F_1 together form a couple of magnitude $F \times a$. This couple tends to rotate the shaft about the point O in a plane containing the shaft and the forces F and F_1; a line drawn through O perpendicular to

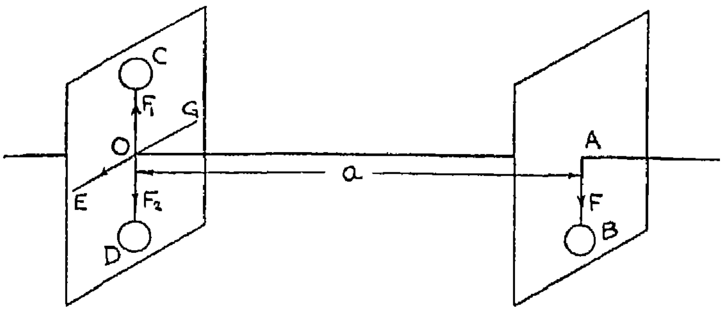


Fig. 150.

the shaft OA and perpendicular to F_1 is thus an axis of rotation of the couple. This line is represented by EOG. Now a rotation can either be clockwise or counter-clockwise and a couple may be conveniently represented by means of a vector drawn along the axis of the couple. The vector can be drawn in one direction along the axis to represent a clockwise couple and in the opposite direction for a counter-clockwise couple. Thus the couple due to F and F_1, which is clockwise, may be represented by a vector OE drawn along an axis of the couple; the length of OE can represent the magnitude of the couple. To represent a counter-clockwise couple a vector OG, opposite in direction to OE, could be drawn. The vector OE is drawn in the plane through O and perpendicular to the shaft.

Looking on an end view of the shaft the vector OE, representing the couple due to F and F_1, is drawn perpendicular to F and F_1. Similarly, a couple tending to cause counter-clockwise movement about O could be represented by a vector

along OG. In balancing problems it is usual, for convenience of drawing, to turn these vectors through 90 degrees. Hence the vector representing the couple due to F_1 and F can be drawn parallel to F but in the plane through O.

§ **189. Several Masses in Different Planes.** In Fig. 151, let OA represent a shaft; W_1, W_2, and W_3 masses attached to the shaft at radii r_1, r_2, and r_3 respectively. The masses W_1,

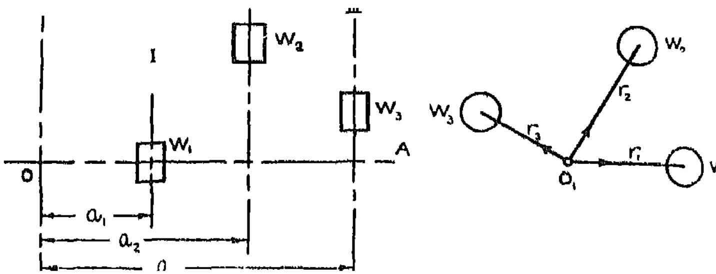
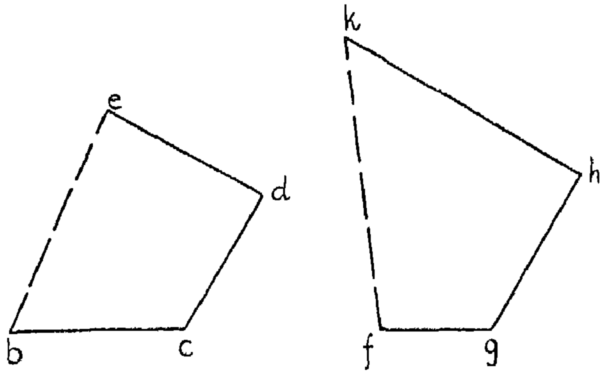


Fig. 151.

W_2, and W_3 rotate in planes I, II, and III respectively. Let a reference plane be chosen at O and the distances of the planes from O be a_1, a_2, and a_3. The forces acting on the shaft are proportional to $W_1 r_1$ in plane I, $W_2 r_2$ in plane II, and $W_3 r_3$ in plane III. The force proportional to $W_1 r_1$ acts in a direction from O_1 to W_1, $W_2 r_2$ from O_1 to W_2, and $W_3 r_3$ from O_1 to W_3. If the masses are transferred to the reference plane, these forces act in the reference plane, and by drawing a vector diagram the unbalanced force can be found. In Fig. 152 (a), bc is drawn parallel to $O_1 W_1$ (Fig. 151) and proportional to $W_1 r_1$, cd parallel to $O_1 W_2$ and proportional to $W_2 r_2$, and de parallel to $O_1 W_3$ and proportional to $W_3 r_3$. The vector be is the sum of the vectors and is proportional to the unbalanced force acting on the shaft.

The couples introduced by transferring the masses to the reference plane are proportional to $W_1 r_1 a_1$, $W_2 r_2 a_2$, and $W_3 r_3 a_3$. A couple may be represented by a vector drawn perpendicular to the plane of the couple. The couple introduced

by transferring W_1 to the reference plane through O is proportional to $W_1 r_1 a_1$ and acts in a plane through $O_1 W_1$ and perpendicular to the paper. The vector representing this couple should be drawn in the plane of the paper and perpendicular to $O_1 W_1$. Similarly the vectors representing the couples $W_2 r_2 a_2$ and $W_3 r_3 a_3$ should be drawn perpendicular to $O_1 W_2$ and $O_1 W_3$ respectively and in the plane of the paper. It



(a) Fig. 152. (b)

is usual, however, to turn these vectors through a right angle for convenience of drawing. The relative inclinations are not affected if each is turned through a right angle. Actually, then, the vectors representing the couples are drawn parallel to $O_1 W_1$, $O_1 W_2$, and $O_1 W_3$ respectively. In Fig. 152 (*b*), *fg* is drawn parallel to $O_1 W_1$ (Fig. 151) and proportional to $W_1 r_1 a_1$, *gh* is drawn parallel to $O_1 W_2$ and proportional to $W_2 r_2 a_2$, and *hk* is drawn parallel to $O_1 W_3$ and proportional to $W_3 r_3 a_3$. The vector *fk* is the sum of the vectors and is proportional to the unbalanced couple acting about O. In Fig. 152 (*a*), *be* is proportional to the resultant force, hence *eb* is proportional to the equilibrant; similarly *fk* represents the resultant couple and *kf* the equilibrant couple.

For complete equilibrium of the shaft the resultant force and resultant couple must be each equal to zero. Expressed mathematically,

$$\sum Wr = 0$$

and

$$\sum Wra = 0.$$

A mass placed in the reference plane such that its Wr is proportional to eb will balance all the forces in the reference plane, but the couples are not balanced. Hence, in general, two planes are required to balance a system of revolving masses.

The formation of a table showing values of Wr and of Wra and the judicious choice of reference plane facilitate the solution of problems on balancing.

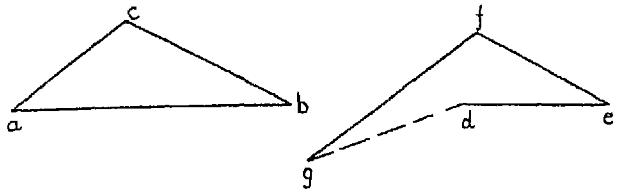
EXAMPLE 1. Four masses A, B, C, and D, weighing 80, 100, 120, and W lb. respectively, are rigidly connected to a shaft at radii 15, 12, 14, and 12 in. respectively from the axis. The shaft revolves about its axis and the planes of revolution of the masses are at equal intervals apart. Determine W and the angular positions of B, C, and D in relation to that of A, in order that the masses may completely balance one another. [*Lond. B.Sc.*]

The planes in which the masses A, B, C, and D are rotating may be represented by the letters A, B, C, and D. Let a = distance between two consecutive planes. Forming a table to show the values of Wr and Wra, these values enable the vector diagrams

Plane	*Mass W*	*Radius r*	*Wr*	*a*	*Wra*	
A	80	15	1,200	$3a$	$3{,}600a$	
B	100	12	1,200	$2a$	$2{,}400a$	
C	120	14	1,680	a	$1{,}680a$	
D	W	12	$12W$	0	0	D is chosen as reference plane

of forces and couples to be drawn. In many cases it will be found advisable to draw the vector diagram of couples first. Thus, in the table, if D is chosen as the reference plane the couple due to the unknown mass W is zero and the other couples are respectively $3{,}600a$, $2{,}400a$, and $1{,}680a$ for the masses A, B, and C respectively. The vector diagram thus becomes a triangle. In Fig. 153 (a), ab is drawn to represent $3{,}600a$, and circular arcs are drawn with radii proportional to $2{,}400a$ and $1{,}680a$, giving a point c. The vector bc represents the couple due to the mass B and ca the couple due to the mass C. The triangle is closed since the resultant couple must be zero. These vectors indicate the relative positions A, B, and C may occupy. In Fig. 154, O_1A is drawn parallel to ab, O_1B parallel to bc, and O_1C parallel to ca. These are the relative angular positions the masses A, B, and C occupy.

In Fig. 153 (*b*) the vector diagram of the forces in the reference plane is drawn. The vector *de* is parallel to *ab* and proportional to 1,200, *ef* is parallel to *bc* and proportional to 1,200, *fg* is parallel to *ca* and proportional to 1,680. The closing vector *gd* represents



(*a*) Fig. 153. (*b*)

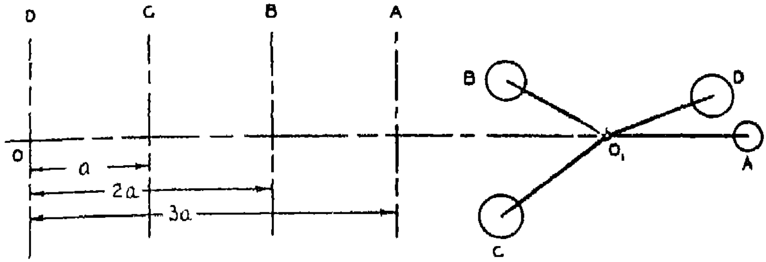


Fig. 154.

the force necessary for equilibrium and to scale represents 1,395, which is proportional to $12W$ in the table.

$$\therefore \; W = \frac{1,395}{12} = 116 \text{ lb.}$$

In Fig. 154, O_1D is drawn parallel to *gd* and is the position the mass W occupies. The relative angular positions of B, C, and D measured from A in a counter-clockwise direction are 157·5°, 213·2°, and 19·5° respectively.

§ **190. Reciprocating Masses.** In an ordinary reciprocating engine, the acceleration of the reciprocating parts is very approximately $\omega^2 r\left(\cos\theta+\frac{r}{l}\cos 2\theta\right)$, as shown in § 66, and the force required to accelerate them is

$$\frac{W}{g}\omega^2 r\left(\cos\theta+\frac{r}{l}\cos 2\theta\right).$$

This expression may be split up into two expressions, viz.

$\frac{W}{g}\omega^2 r \cos\theta$ and $\frac{W}{g}\frac{\omega^2 r^2}{l}\cos 2\theta$. The former is the primary accelerating force and the latter the secondary accelerating force. The secondary force is smaller than the primary, and for moderate speeds it is usual to neglect the secondary effects. The secondary force has, however, twice the frequency of the primary force, and in high-speed engines this is of considerable importance. To keep the problems in their simplest form, only primary forces will be dealt with.

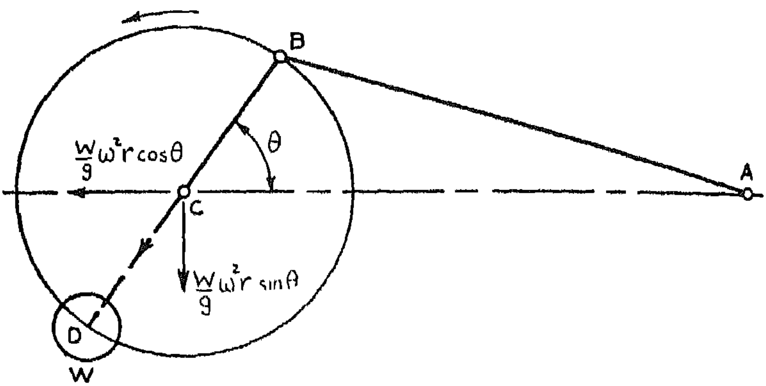


Fig. 155.

In Fig. 155, let CB represent the crank and BA the connecting-rod. Assume the reciprocating parts are concentrated at A and of weight W. For the given configuration the acceleration of A is positive, that is the speed is increasing and the primary force $\frac{W}{g}\omega^2 r\cos\theta$ required to accelerate W must act in a direction from A to C. The reaction of this force upon the frame of the engine is in the opposite direction and this reaction tends to move the frame in a direction from C to A. Any attempt at balance must be to balance this reaction. Suppose a mass of weight W is placed at crank radius diametrically opposite the crank pin. This mass is shown at D. The disturbing effect of this mass upon the frame of the engine is $\frac{W}{g}\omega^2 r$. This can be split up into two components

acting through the centre of the crank-shaft of $\frac{W}{g}\omega^2 r\cos\theta$ acting horizontally and $\frac{W}{g}\omega^2 r\sin\theta$ acting vertically. The horizontal component balances the primary force $\frac{W}{g}\omega^2 r\cos\theta$ acting on the frame in the opposite direction, and the mass W placed diametrically opposite the crank pin has thus effected primary balance in a horizontal direction. There is, however, due to W at D, a vertical component of $\frac{W}{g}\omega^2 r$ which in magnitude is $\frac{W}{g}\omega^2 r\sin\theta$, and this is unbalanced. The maximum value of this component is $\frac{W}{g}\omega^2 r$ and occurs when $\sin\theta = 1$, or when $\theta = 90°$. The introduction of the mass W at D balances the horizontal forces but induces a vertical unbalanced force equal in magnitude to the original horizontal unbalanced force.

In practice a compromise is made by placing $\frac{2}{3}W$ at crank radius. The unbalanced horizontal force is then $\frac{1}{3}\frac{W}{g}\omega^2 r\cos\theta$ and the unbalanced vertical force is $\frac{2}{3}\frac{W}{g}\omega^2 r\sin\theta$. The usual practical method is to consider $\frac{2}{3}$ of the reciprocating mass transferred to the crank pin and balance in the usual manner.

Let W_1 = weight of revolving masses at crank pin,

W_2 = weight of reciprocating masses.

Equivalent mass at crank pin $W = W_1 + \frac{2}{3}W_2$.

EXAMPLE 2. A single-cylinder horizontal engine runs at 120 revs. per min. The stroke is $1\frac{1}{2}$ ft., mass of revolving parts assumed concentrated at the crank-pin is 200 lb., and the mass of the reciprocating parts is 300 lb. Determine the magnitude of the balance weights required, assuming that their centres of gravity are 7 in. from the centre of the crank-shaft. Calculate the maximum value of the unbalanced horizontal force introduced by the balance weight.

Equivalent mass at crank pin $W = 200 + \frac{2}{3}\times 300 = 400$ lb.

Let W_3 = balance weight required:

then $$\frac{W_3}{g} \times \frac{7}{12}\omega^2 = \frac{400}{g} \times \frac{9}{12} \times \omega^2,$$

$$\therefore\ W_3 = 514 \text{ lb.}$$

Maximum unbalanced horizontal force

$$= \frac{1}{3}\frac{W_2}{g} \times \frac{9}{12} \times \omega^2$$

$$= \frac{1}{3} \cdot \frac{300}{32{\cdot}2} \times \frac{9}{12} \times \left(\frac{120}{60} \times 2\pi\right)^2$$

$$= 368 \text{ lb.}$$

§ **191. Locomotive Engines.** The usual type of locomotive in this country has two cylinders. These may be placed either inside or outside the front bogie wheels. The engines are known as inside and outside cylinder locomotives respectively. The two cranks are usually placed at right angles to each other and balance weights are placed in the driving-wheels. To increase the adhesion between the locomotive and the rails, two or three pairs of driving-wheels may be coupled together and the engine is known as a coupled locomotive. An uncoupled locomotive has one pair of driving-wheels, and as far as the balancing problem is concerned there are four planes in which masses are rotating, viz. the two planes of the driving-wheels and the two planes containing the crank pins. In a coupled locomotive the number of planes is six, since the coupling rods are rotating in planes outside the driving-wheels. It is not desirable here to enter into a complete discussion of the balancing of coupled locomotives, and to keep the problem in its simplest form the balance of uncoupled locomotives only will be dealt with, although this type of locomotive is rarely seen in modern railway stock.

The revolving masses at the crank pin may be completely balanced, but the reciprocating masses cannot be balanced completely—only partially. The unbalanced horizontal force causes a variation of the tractive force, in addition to a swaying couple. The unbalanced vertical forces cause the pressure between the driving-wheels and rails to vary; this variation is known as hammer blow.

EXAMPLE 3. Find the balance weights required to balance all the revolving masses and two-thirds of the reciprocating masses of an inside-cylinder uncoupled locomotive. The cylinders are 2 ft. apart and the balance planes 5 ft. apart. Mass of revolving parts per cylinder 700 lb., reciprocating parts per cylinder 500 lb. Crank radius is 1 ft., and the mass centres of the balance weights are at a radius of 30 in.

Equivalent mass at crank radius $= 700+\frac{2}{3}\times 500 = 1{,}033$ lb. The table is compiled to assist the drawing of the vector diagrams.

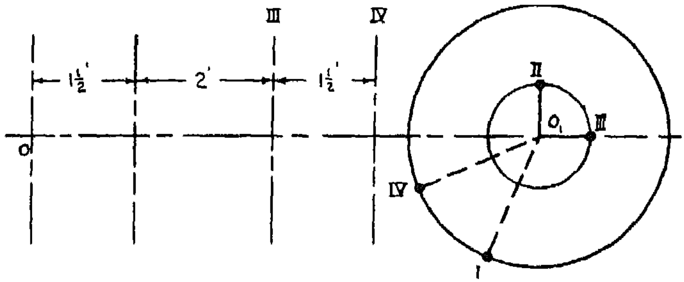


Fig. 156.

In Fig. 156 the distances apart of the planes is shown and the position of the cranks in end view. Plane I is chosen as reference plane and the vector sum of the *Wra* column must be zero. In Fig. 157 (*a*), *ab* is drawn parallel to the crank in plane III and

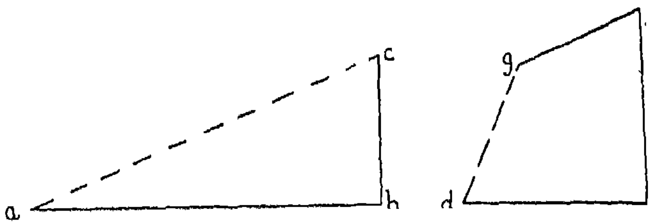
Plane	*Mass W*	*r*	*Wr*	*a*	*Wra*	
I	..	$2\frac{1}{2}$	..	0	0	Plane I chosen as reference plane
II	1,033	1	1,033	$1\frac{1}{2}$	1,550	
III	1,033	1	1,033	$3\frac{1}{2}$	3,620	
IV	*W*	$2\frac{1}{2}$	$2\frac{1}{2}W$	5	$12\frac{1}{2}W$	

proportional to 3,620; *bc* is drawn parallel to the crank in plane II and proportional to 1,550. The closing vector *ca* represents the *Wra* for plane IV. *ca* scales 3,940,

$$\therefore\ 12\tfrac{1}{2}W = 3{,}940,$$
$$W = 315 \text{ lb.}$$

In Fig. 157 (*b*), *de* is drawn parallel to *ab* and proportional to 1,033, *ef* is parallel to *bc* and proportional to 1,033, *fg* is parallel to *ca* and proportional to $2\frac{1}{2}W$ or 788. The closing vector is the

Wr for plane I and this scales 778, giving the balance mass in plane I as 315 lb. The angular positions of the balance masses relative to the cranks is shown in Fig. 156, in which a line parallel



(a) Fig. 157. (b)

to ca (Fig. 157) gives the angular position of the balance mass in plane IV and a line parallel to gd gives the angular position of the balance mass in plane I. The inclination of these two planes with their adjacent cranks is the same and the two balance masses are equal.

§192. Unbalanced Forces of Partially Balanced Locomotives. As already explained, for two-cylinder locomotive engines with cranks at right angles, the usual practice is to balance completely the revolving masses and to balance partially the reciprocating masses.

Let W_1 = weight of revolving parts per cylinder,

W_2 = weight of reciprocating parts per cylinder,

W_3 = weight of each balancing mass at crank radius,

w_1 = portion of the balancing mass due to W_1,

w_2 = portion of the balancing mass due to W_2.

Then,
$$W_3 = w_1 + w_2,$$
$$w_2 = \frac{\frac{2}{3}W_2}{\frac{2}{3}W_2+W_1} \times W_3,$$
and
$$w_1 = \frac{W_1}{\frac{2}{3}W_2+W_1} \times W_3.$$

Now, w_1 is the portion of the balancing mass required to balance the revolving parts, and as these are completely balanced, there are no unbalanced forces due to w_1. Since w_2 only partially balances the reciprocating masses W_2, the

unbalanced forces are due to the unbalanced part of W_2 acting horizontally and the vertical component of w_2.

For each cylinder the unbalanced horizontal force due to $\frac{W_2}{3}$ unbalanced is $\frac{W_2}{3g}\omega^2 r\cos\theta$. If the leading crank makes an angle θ with the inner dead centre, the other crank makes an angle of $(\theta-90^\circ)$ and the two unbalanced forces are $\frac{W_2}{3g}\omega^2 r\cos\theta$ and $\frac{W_2}{3g}\omega^2 r\cos(\theta-90^\circ)$. The total unbalanced force is the sum of these two quantities, viz.

$$\frac{W_2}{3g}\omega^2 r[\cos\theta+\cos(\theta-90^\circ)]$$

or

$$\frac{W_2}{3g}\omega^2 r(\cos\theta+\sin\theta).$$

This is a maximum when $\theta = 45^\circ$ or $22'\,^\circ$, and the maximum unbalanced horizontal force is $\frac{\sqrt{2}W_2}{3g}\omega' r$. This maximum unbalanced force causes a variation of draw-bar pull.

The two unbalanced horizontal forces of $\frac{W_2}{3g}w^2 r\cos\theta$ and $\frac{W_2}{3g}w^2 r\cos(\theta-90)$, for crank angles of θ and $(\theta-90)$ respectively, vary in magnitude and direction according to the value of θ. For values of θ between 0° and 90° these two forces are in the same direction, but except when $\theta=45^\circ$, they differ in magnitude. For values of θ between 90° and 180° they are in opposite directions and their resultant, except when $\theta=135^\circ$, is an unbalanced horizontal force which has an unbalanced moment about the centre line between the two cylinders. When $\theta=135^\circ$, and also when $\theta=315^\circ$, this effect reduces to an unbalanced pure couple, the forces being equal, each of magnitude $\frac{W_2}{3g}w^2 r\times\frac{1}{\sqrt{2}}$, and in opposite directions. This couple has a swaying effect about a vertical axis and tends to sway the engine alternately in clockwise and counter-clockwise directions; it is consequently referred

$$v = r_1\omega = 3\tfrac{1}{4}\times 47\cdot 7 = 155 \text{ ft. per sec.}$$
$$= 105\cdot 7 \text{ miles per hour.}$$

EXERCISES. XII

1. Two masses, of 20 lb. and 40 lb. respectively, are attached to a balanced disk at an angular distance apart of 90°, and at radii 2 and 3 ft. respectively. Find the resultant force on the axis when the speed of the disk is 200 revs. per min., and determine the angular position and magnitude of a mass placed at $2\frac{1}{2}$ ft. radius which will make the force on the axis zero at all speeds.

2. Four masses of magnitude 10, 12, and 14 lb. and W lb. revolve in planes A, B, C, and D respectively. The planes are spaced A to B 2 ft., A to C 3 ft., A to D 5 ft. The masses are all at the same radius. Find the magnitude of W and the relative angular positions of the masses for complete balance.

3. In a three-crank vertical engine the weight of the moving parts on each crank is equal. Show that it is balanced for the vertical components of the inertia forces, but not for the couple.

4. A single-cylinder vertical engine with a stroke of 18 in. runs at 250 revs. per min. The reciprocating parts weigh 120 lb. and may be assumed to have simple harmonic motion; the revolving parts are equivalent to a mass of 70 lb. at a radius of 9 in. Determine the balancing mass, to be placed opposite the crank and at a radius of 15 in., which is equivalent to all the revolving and two-thirds of the reciprocating masses.

Find also the magnitude of the remaining unbalanced force when the crank has turned through an angle of 30° from a dead centre.
[*Inst. C. E.*]

5. Three masses A, B, and C, weighing 2 lb., 6 lb., and 5 lb. respectively, are rigidly attached to a disk at radii of 10 in., 5 in., and 9 in. respectively. The angles between A and B, B and C, and C and A are 60°, 165°, and 135° respectively. A fourth mass, D, is attached to the disk at a radius of 4 in. Determine the magnitude and angular position of D so that the four masses may completely balance one another when the disk is rotated.

Explain why, for complete balance, mass D must be placed in the same plane as the masses A, B, and C. [*Inst. C. E.*]

6. What is meant by the 'hammer blow' of a locomotive? When the revolving parts of a locomotive are completely balanced, show how the magnitude of the hammer blow at a given speed can be varied by an alteration in the proportion of the weight of the reciprocating parts treated as a revolving mass, when the requisite balancing masses have been added.

7. Find the balance weights required for an inside-cylinder uncoupled locomotive when running under the following conditions:

Weight of reciprocating parts (per cylinder) . .	= 550 lb.
Weight of rotating parts (per cylinder) . . .	= 650 lb.
Stroke	= 24 in.
Cylinder centres	= 25 in.
Wheel centres	= 60 in.
Radius of balance weights	= 30 in.

Balance the whole of the rotating and two-thirds of the reciprocating parts. [*Inst. C. E.*]

8. The cylinders of a four-cylinder vertical marine engine are pitched at equal distances apart. The reciprocating masses belonging to each of the two middle lines of parts weigh 1 ton. The cranks of the middle pair of cylinders are set at an angle of 100°. Find the angle between each outside crank and its neighbour, and the weight of each of the outer reciprocating lines of parts so that the engine shall be in balance for primary forces and couples. [*Lond. B.Sc.*]

9. The horizontal distance between the centre lines of the cylinders of a locomotive is 2 ft., and the horizontal distance between the planes in which the mass centres of the balance weights in the wheels revolve is 5 ft. The cranks are at right angles and are each 1 ft. radius. The reciprocating masses weigh 600 lb. per cylinder, two-thirds of which is balanced by balance weights placed in the driving-wheels. The load on each driving-wheel is 7·5 tons. Calculate the maximum and minimum pressure between the wheel and the rail when the engine runs at 80 miles per hour. Diameter of driving-wheels 7 ft. [*Inst. C. E.*]

10. An outside-cylinder uncoupled locomotive has reciprocating parts which weigh 500 lb. per cylinder, revolving parts which weigh 600 lb. per cylinder, a stroke of 24 in., and driving-wheels 7 ft. in diameter. When the whole of the rotating parts and two-thirds of the reciprocating parts are balanced, the equivalent balancing weight at crank radius in each wheel is 1,022 lb. Determine the maximum variation in wheel pressure and the maximum variation in tractive effort due to the unbalanced masses when the locomotive has a speed of 60 miles per hour. [*Inst. C. E.*]

11. A shaft carries four revolving masses, A, B, C, and D, in this order along the axis. The mass A may be assumed concentrated at a radius of 12 in., B at 15 in., C at 14 in., and D at 18 in. The weights of A, C, and D are 15 lb., 10 lb., and 8 lb. respectively. The planes of revolution of A and B are 15 in. apart, and of B and C 18 in. apart. The angle between the masses A and C is 90°. Determine (a) the angles between the masses A, B, and D, (b) the distance between the planes of revolution of C and D, and (c) the weight of the mass B, so that the shaft may be in perfect balance. [*Inst. C. E.*]

12. What are the necessary conditions for the complete balance of

a system of masses revolving about a common axis? Carefully explain how you would balance a number of masses revolving about the same axis but in different planes. [*I. Mech. E.*]

13. A system of revolving masses rotate in three planes 1, 2, and 3, placed in this order. Masses of 200 lb. revolve in planes 1 and 3 at a crank radius of 18 in., the cranks being at right angles, and a mass of 300 lb. revolves in plane 2 at a radius of 18 in., and the crank is inclined 135° to cranks 1 and 2. Two planes A and B are between planes 1 and 2 and between 2 and 3 respectively. The distances between the planes are $1A = 1$ ft., $A2 = 2$ ft., $2B = 2$ ft., $B3 = 1$ ft. Find the magnitude of the unbalanced couple when the system revolves at 300 revs. per min.

Determine the magnitude and position of masses which must be introduced in planes A and B at a radius of 2 ft. so as to give perfect balance to this revolving system. [*I. Mech. E.*]

14. A twin-cylinder Vee engine has cylinders at 90° which drive on to a single crank. The weight of the reciprocating parts of each cylinder, including one-third of the weight of the connecting-rod, is 15 lb. The weight of the crank pin and webs plus two-thirds of the weight of the connecting-rod is equivalent to 30 lb. at the crank pin. The stroke is 6 in. Determine the primary unbalanced force at 1,000 revs. per min. [*Lond. B.Sc.*]

15. Five pulleys 2 ft. apart are mounted on a shaft 12 ft. between bearings. These are out of balance to the following extent:

No. 1,	2·0 lb. at 2·5 ft.	radius.
No. 2,	4·0 lb. at 3·0 ft.	„
No. 3,	3·5 lb. at 2·0 ft.	„
No. 4,	2·0 lb. at 2·0 ft.	„
No. 5,	2·0 lb. at 3·0 ft.	„

If $\theta_1 = 0°$, $\theta_2 = 45°$, $\theta_3 = 90°$, $\theta_4 = 120°$, $\theta_5 = 240°$ define the angular positions of the radii of the centre of gravity, determine the two masses which will balance the system when placed on Nos. 2 and 4 pulleys, at 3 ft. radius in No. 2 and 2 ft. radius in No. 4. [*Lond. B.Sc.*]

16. A casting under process of machining weighs 450 lb. and is bolted to the face plate of a lathe. Its centre of gravity G (Fig. 158) is 9 in. from the lathe axis and 14 in. from the face plate.

The workman obtains static balance by means of two weights bolted to the face plate; one of these weighs 90 lb. and is fixed in the position shown in the end elevation (Fig. 158); its centre of gravity is 5 in. from the face plate. The other weighs 80 lb. and has its centre of gravity 4 in. from the face plate.

Find the position of the centre of gravity of the second balance weight. Find also the magnitude of the rocking couple when the lathe is run at 60 revs. per min. [*Lond. B.Sc.*]

17. From the data given below, determine the magnitude and

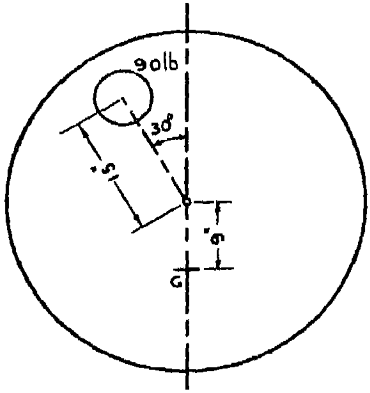


Fig. 158.

position of the balance weights to balance the whole of the rotating masses and two-thirds of the reciprocating masses of an outside-cylinder uncoupled locomotive.

Weight of reciprocating masses for each cylinder .	= 480 lb.
Weight of rotating masses for each cylinder . .	= 560 lb.
Distance between cylinder centres	= 65 in.
Distance between planes in which the mass centres of the balance weights rotate	= 55 in.
Stroke	= 24 in.
Radius of mass centres of balance weights . .	= 30 in.

[*Lond. B.Sc.*]

18. A four-crank marine engine (Fig. 159) runs at 85 revs. per min. and has a stroke of 42 in. The weights of the reciprocating parts are: H.P. = 900 lb., $I.P._1$ = 1,050 lb., and $I.P._2$ = 1,250 lb. Determine the weight of the L.P. reciprocating parts, and the crank angles so that the reciprocating parts are in complete primary balance.

[*Lond. B.Sc.*]

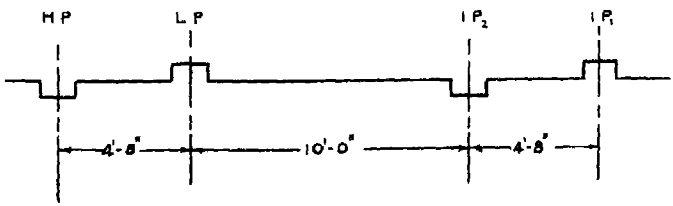


Fig. 159.

19. A two-cylinder inside engine has its cylinders 23 in. apart, and the balance weights are in planes 60 in. apart, the planes being symmetrically placed about the engine centre line. For each cylinder the revolving masses are 600 lb. at the crank-pin radius (13 in.), and

the reciprocating parts are 570 lb. All the revolving and two-thirds of the reciprocating masses are balanced. The driving-wheels are 6 ft. in diameter. When the engine runs at 40 miles per hour, find (*a*) the hammer blow; (*b*) the variation in tractive force; (*c*) the swaying couple. [*Lond. B.Sc.*]

20. In a four-cranked engine the distances between the cylinders are all equal. Three reciprocating masses reckoned in succession from the right are $1\frac{1}{2}$, 2, and $2\frac{1}{2}$ tons. Find the fourth reciprocating mass, and the crank angles so that the reciprocating masses mutually balance. [*Lond. B.Sc.*]

21. Four masses A, B, C, and D, in different planes, are to be arranged in this order along a shaft to give complete balance when rotating. The planes containing the masses B and C are 1 ft. apart. The masses B and C are at right angles to each other and the mass D is at an angle of 135 degrees to both B and C. Find where the planes containing the masses A and D should be placed and the magnitude and angular position of the mass A.

Mass	*Weight (lb.)*	*Radius (in.)*
A	W	1·5
B	4	2·5
C	6	1·0
D	5	1·25

[*Lond. B.Sc.*]

22. Masses of 10, 12, and 8 lb. rotate respectively in planes A, B, and C spaced 1 ft. apart; the centres of gravity of these masses are 3, 4, and 5 in. respectively from the axis of rotation. Find the angular positions of these masses so that the dynamic forces balance. What is the magnitude of the unbalanced couple when the speed of rotation is 250 revolutions per minute?

Find the magnitudes of weights to be placed in two planes situated $1\frac{1}{2}$ ft. on either side of plane B to balance this couple and also to maintain dynamic balance when the centre of gravity of each weight is 3 in. from the axis. [*Lond. B.Sc.*]

23. Details of masses carried by a rotating shaft are as follows:

Plane	*Mass* (W) lb.	*Radius* (r) *ft.*
A	20	1
B	W	1
C	15	1
D	12	1

The planes A and C are $1\frac{1}{2}$ ft. on either side of the plane B. Find the position of the plane D, the magnitude of W and its angular position relative to the other masses when the angle between the masses in planes A and C is 120° and the masses are balanced. [*Lond. B.Sc.*]

CHAPTER XIII

CAMS

§ **193. Simple Rotating Cam.** Cams are used to impart motion to a follower; the motion is usually of an intermittent nature and may be reciprocating or oscillating. Cams usually

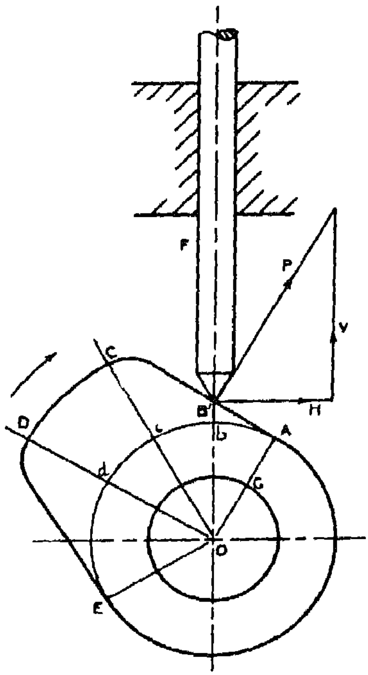


Fig. 160.

have a rotary motion at constant speed, but occasionally have rectilinear motion. A simple type of rotating cam is shown in Fig. 160 in which F is the follower whose line of motion passes through the centre of the cam, and for simplicity the contact between the cam and follower is point or line contact at B. In practice it is usual to round off the end of the follower or to provide a roller to reduce the wear of the cam

surface. The effect of a roller will be considered later. The follower F is constrained by guides to move in a vertical direction, either up or down. Assuming the cam to rotate in a clockwise direction, the follower is lifting while the cam moves through the angle AOC and it is at rest while the cam moves through the angle COD, and it falls while the cam moves through the angle DOE.

The angle AOC is the angle of lift.

The angle DOE is the angle of fall.

The angle AOE is the angle of cam action.

The radius of the cam shaft is OG, GA is the least thickness of metal of the cam, and OA is the least radius.

The lift of the follower $= cC = dD$ in this particular case. The distance cC or dD is also known as the height of the cam, but the height of the cam is only equal to the lift of the follower when the line of stroke passes through the centre of the cam. To determine whether the outline of cam is suitable for producing a given lift of the follower while the cam moves through the angle AOC, it is necessary to construct a displacement diagram, and from this to deduce the velocity and acceleration of the follower. The acceleration of the follower determines the accelerating force required to accelerate it, and this should either be fairly uniform for each half of the lift or should vary in a gradual manner. In general, the accelerating force should be kept as small as possible. A further point to be noted is that the lateral pressure of the cam on the follower should be within reasonable limits. Neglecting the small amount of friction between the follower and the cam, the pressure which the cam exerts on the follower is normal to the surface of the cam at the point of contact. Let $P =$ normal pressure. This has two components, V vertical and H horizontal; the vertical component is the force available for lifting and accelerating the follower, the horizontal component merely causes a side pressure on the guides with consequent loss due to friction. Excessive side pressure will result in wear, and in time the follower becomes slack in the guides and objectionable noise and rattle occurs.

§ 194. **Displacement Diagram.** When the cam is in the position shown in Fig. 160, the displacement of the follower is bB. Radial distances measured from the least radius

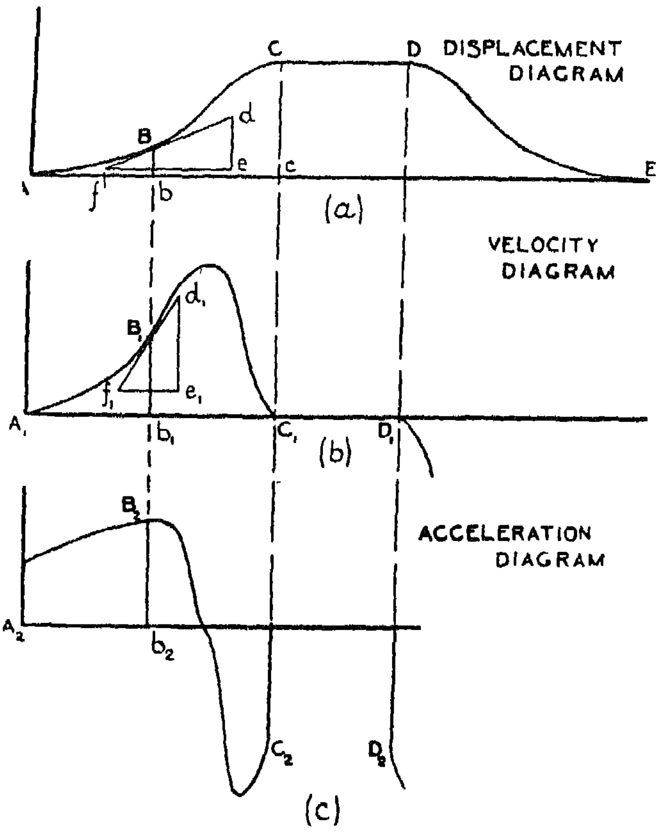


Fig. 161.

$AbcdE$ give the displacements for corresponding cam angular movements. The outline $ACDEdcbA$ may be regarded as a polar diagram of displacements. To construct a rectangular diagram of displacements, values of the displacement are plotted vertically against the angular movement of the cam plotted horizontally. In Fig. 161 (a), AE is proportional to

the angle of cam action, Ab is proportional to the angle AOB (Fig. 160). The ordinate bB, Fig. 161 (*a*), is made equal to or proportional to the displacement bB in Fig. 160. Plotting other values and drawing a smooth curve through the points obtained, the displacement curve $ABCDE$, Fig. 161 (*a*), is obtained.

§ **195. Velocity Diagram.** If the cam rotates at constant angular speed, the angular movement of the cam is proportional to the time, and since $\frac{ds}{dt}$ or rate of change of displacement is velocity, the slope of the displacement curve at any point is proportional to the velocity of the follower corresponding to that position of the cam. Taking the point B on the displacement curve, Fig. 161 (*a*), the tangent fd is drawn, the slope of the curve is $\frac{de}{ef}$ which is proportional to the velocity of the follower. On Fig. 161 (*b*), $b_1 B_1$ is drawn proportional to $\frac{de}{ef}$ for the given angular position of the cam. Taking other points on the displacement curve and drawing tangents to find the slope, and plotting these values, the velocity diagram $A_1 B_1 C_1$ is obtained; this is the velocity diagram while the follower is being lifted. The diagram for the fall of the follower is obtained in a similar manner but is not shown.

§ **196. Acceleration Diagram.** Acceleration is rate of change of velocity with respect to time or $\frac{dv}{dt}$, and an acceleration diagram is deduced from the velocity diagram in the same way that the velocity diagram is deduced from the displacement diagram. The tangent $f_1 d_1$ at B_1, Fig. 161 (*b*), is drawn and the slope of the velocity curve at this point is $\frac{d_1 e_1}{e_1 f_1}$. This quantity is proportional to the acceleration of the follower for this position of the cam. In Fig. 161 (*c*), $b_2 B_2$ is plotted proportional to $\frac{d_1 e_1}{e_1 f_1}$, and repeating the

process for other points, the acceleration curve $B_2 C_2 D_2$ is obtained. This represents the acceleration of the follower while it is being lifted; the acceleration diagram while falling is obtained in a similar manner. The acceleration diagram is not of good shape and the given cam profile would cause irregular accelerating force.

§ **197. Simple Harmonic Motion of Follower.** The acceleration of a follower having simple harmonic motion varies as the distance from the centre position, hence the accelerating force is a maximum when the follower commences its lift and is a maximum negative value at the end

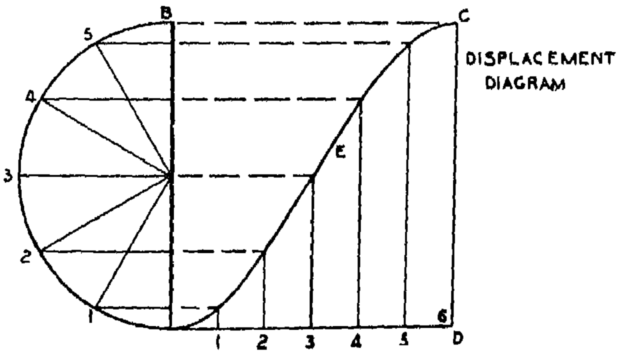


Fig. 162.

of the lift. The acceleration diagram consists of two triangles and the acceleration changes uniformly. The displacement diagram for the lift of the follower is a double sine curve as shown in Fig. 162, in which AD corresponds to the angle of lift of the cam and CD is equal to or proportional to the lift of the follower. The displacement curve is obtained by dividing the angle of lift into a number of equal parts, six in this case, and on AB, which is equal to DC, a semicircle is drawn and divided into an equal number of parts around the circumference. By projecting corresponding points along to meet the verticals through the equal divisions, points on the curve AEC are obtained.

Another form of displacement curve sometimes used is that of a double parabola. This curve gives a uniform acceleration

for the first half of the lift and a uniform deceleration for the second half. In Fig. 163, AD is made proportional to the angle of lift, and $DC = AB$ is equal to or proportional to the lift. AD is divided into a number of equal parts, the mid ordinate through the division 3 is also divided into the same number of equal parts. By joining A to a on the vertical through 3, the point at which this line cuts the vertical through division 1 gives a point on the parabola. Alternatively, if G is the mid point of AB, AG is divided in the ratio of 1, 4, and

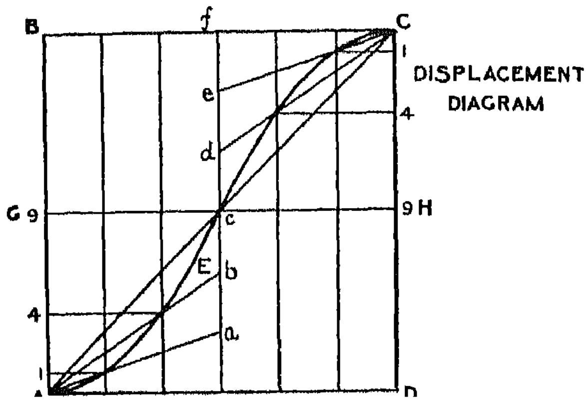


Fig. 163.

9, which are the squares of 1, 2, and 3 respectively. Similarly CH is divided in the ratio of 1, 4, and 9. The curve drawn through the points obtained in either of these ways is the displacement curve and in the figure is AEC.

§ **198. Construction of Cam Profile.** Having given the angle of lift, lift of follower, etc., the method of constructing the cam profile will now be given. To keep the problem in a simple form, point contact will be assumed (i.e. no roller on follower) and the line of stroke of the follower assumed to pass through the centre of the cam. It will only be necessary to describe the procedure for finding the cam profile while lifting the follower, as the profile for determining the fall of the follower is constructed in exactly the same way. Since the motion between the cam and the follower is relative, this

motion is not altered if the cam is assumed to be at rest and the follower to move round the cam in a direction opposite to the motion of the cam. To make a definite problem, let the angle of lift be 60° and the lift $1\frac{1}{4}$ in., the least radius be $1\frac{3}{4}$ in., and the motion of the follower be simple harmonic.

In Fig. 164 a circle is drawn of radius OA which is equal to the least radius of the cam $1\frac{3}{4}$ in. AB is the lift of the follower $1\frac{1}{4}$ in. The angle AOC is set off equal to the angle of lift 60°, and is divided into, say, six equal angles by the

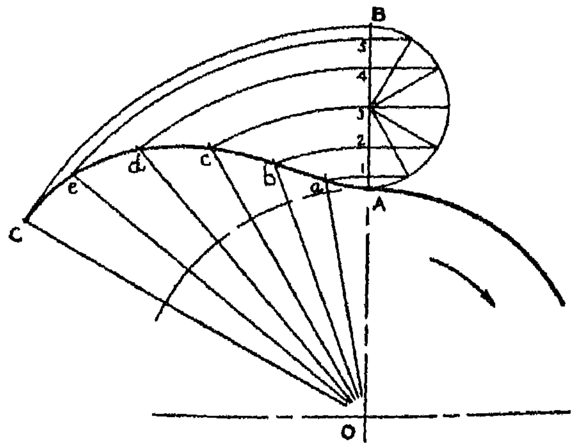


Fig. 164.

radial lines Oa, Ob, etc. The lift AB is divided into six parts corresponding to the displacement diagram. This is conveniently accomplished by drawing a semicircle on AB as diameter and dividing the semicircle into six equal parts. Projecting across on to AB, the points 1, 2, 3, etc., are obtained which are the successive positions of the follower for equiangular movements of the cam. The position 1 is the position the follower occupies when the cam has turned through the angle AOa, position 2 when the cam has turned through the angle AOb, and so on. Swinging round the distance $O1$ to meet Oa gives a point a on the cam profile. Similarly swinging round $O2$ to meet Ob, another point b on the cam profile is obtained. Proceeding in this manner the

points c, d, e, and C are obtained, and a smooth curve drawn through the points gives the required cam profile.

§ 199. **Effect of Roller.** The wear of the face of the cam and the end of the follower may be considerably reduced by placing a roller on the end of the follower. The effect of the roller modifies slightly the shape of the cam, since the point

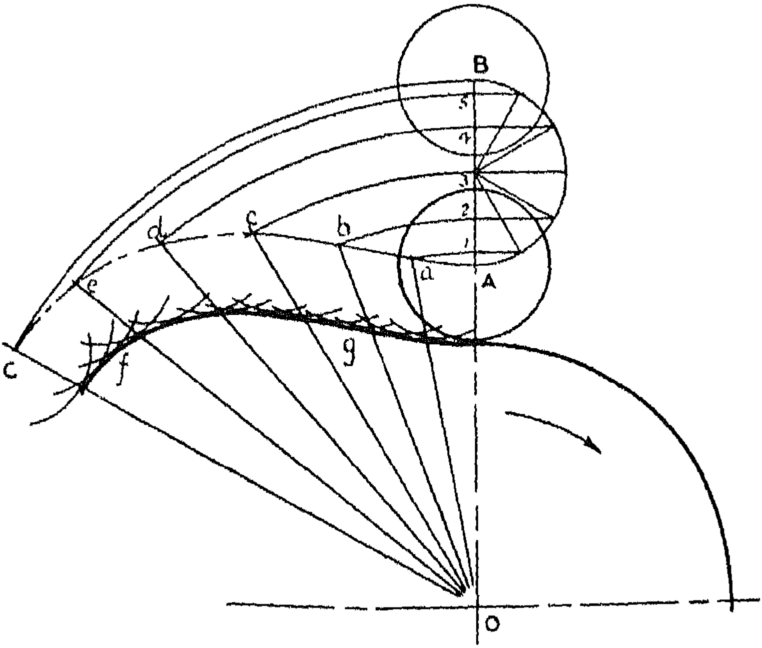


Fig. 165.

of contact between the cam and the roller is not always on the line of stroke of the follower. The motion of the follower is the same as that of the centre of the roller, and before the actual cam profile can be found it is necessary to construct a path which represents the path of the centre of the roller as it moves round the cam. This path is known as the *pitch* profile, and the method of obtaining the pitch profile is similar to that for finding the cam profile, as explained in § 198, with the exception that the commencing radius is increased by the radius of the roller. Take the same problem

as in § 198, with the exception that a roller of 1 in. diameter is fitted to the end of the follower. In Fig. 165 the commencing radius OA is the least radius plus the radius of the roller, i.e. $OA = 1\frac{3}{4} + \frac{1}{2} = 2\frac{1}{4}$ in. AB is the lift of the follower and the points A and B represent the extreme positions of the centre of the roller. The circles drawn with $\frac{1}{2}$ in. radius and centres at A and B represent the positions of the follower at the commencement and end of the lift respectively. The pitch profile $AabcdeC$ is constructed as already explained, and represents the path over which the centre of the roller moves if the follower is assumed to move round the cam in a direction opposite to that of rotation. The cam is assumed to be moving in a clockwise direction. The cam profile is obtained by striking off a series of circular arcs, with centres on the pitch profile, and the envelope of the arcs thus drawn gives the cam profile, which is fg.

§ **200. Line of Stroke of Follower displaced from Cam Centre.** When the line of stroke of the follower does not pass through the centre of the cam, but is displaced, the shape of the cam is further modified and the method of procedure is slightly different. Taking the same problem as in § 198 but with a roller of 1 in. diameter and line of stroke displaced $\frac{3}{4}$ in., the line of stroke is along BD, Fig. 166, where $OD = \frac{3}{4}$ in. The lift of the follower is AB, and OA = least radius + radius of roller, i.e. $1\frac{3}{4} + \frac{1}{2} = 2\frac{1}{4}$ in. The circles whose centres are at A and B, and whose diameters are 1 in., represent the positions of the roller when at the bottom and top of the lift respectively. Assuming the follower to move round the cam, the line of stroke will occupy the positions DA, ha, kb, lc, md, ne, and pC respectively. The angle DOp is 60° and the points h, k, l, etc., divide the circle of radius $OD = \frac{3}{4}$ in. into, say, six equal parts. The lift AB is divided at 1, 2, 3, etc., for simple harmonic motion. Swinging round from centre O with radius $O1$ to cut ha in a, a point a on the pitch profile is obtained. Similarly b is obtained by swinging round $O2$ to kb at b. Proceeding in this manner the points c, d, e, and C are obtained and a smooth curve through all the points gives the pitch profile. Circular arcs, of radius equal

to the radius of the roller, from the pitch profile give the cam profile $EfgH$.

An alternative method is to regard the cam as moving as it does in an actual case. It will be noted that when the centre of the roller is at A, the line joining A and O is not coincident with the line joining O and B, where B is the

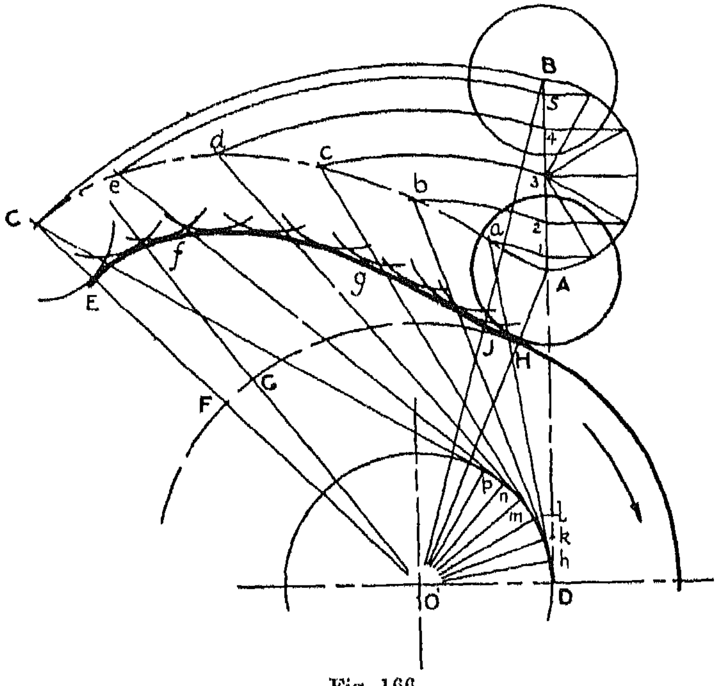


Fig. 166.

centre of the roller in the top position. The angle AOB is the obliquity angle. When the centre of the roller is at A, H is a point on OA and on the least radius of the cam. Similarly J is a point on OB and on the least radius. Let $HOG = 60°$. For the full lift of the follower, G will have moved to H, but a point on OB is J, hence G has moved too far by the amount JH or by the angle JOH = angle AOB. The corrected position for the centre of the roller is along OF, where $FG = JH$, or making the angle GOF equal to AOB, the position of the centre of the roller for the top position is

along OF produced. Swinging round from B with centre O and cutting the line OF produced at C gives the point C on the pitch profile. Repeating this process for intermediate positions of the centre of the roller 1, 2, 3, etc., and taking account of the varying obliquity, the complete pitch profile is obtained.

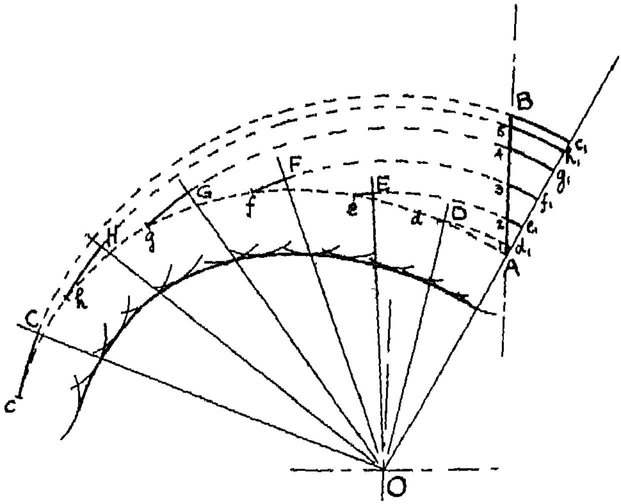


Fig. 167.

Another alternative method is shown in Fig. 167, in which AB represents the lift of the follower and OA is the least radius of the cam plus the radius of the roller. The lift AB is divided at the points 1, 2, 3, 4, and 5 for, say, simple harmonic motion. The angle AOC is the angle of lift and this is divided into six equal divisions, as previously described, by the radial lines OD, OE, OF, OG, and OH. When the centre of the roller moves from A to 1 the cam moves through the angle DOA and consequently D moves to d_1 and has moved too far by the length of the arc $1d_1$. Hence if the arc Dd is set back by an amount equal to $1d_1$, the point d will coincide with 1 when the cam has moved through the angle DOA. The point d is thus the centre of the roller when the follower moves round the cam. Proceeding in this manner the arcs Ee, Ff, Gg, Hh, and Cc are set back by amounts equal to $2e_1$, $3f_1$, $4g_1$, $5h_1$,

and Bc_1 respectively. A smooth curve drawn through the points *chgfed* and A gives the pitch profile, and by drawing a series of circular arcs, of radius equal to the radius of the roller, the envelope of the cam profile is obtained.

§ **201. Follower with Flat Plate.** The follower may be fitted with a flat plate as in Fig. 168. The flat plate A is fixed

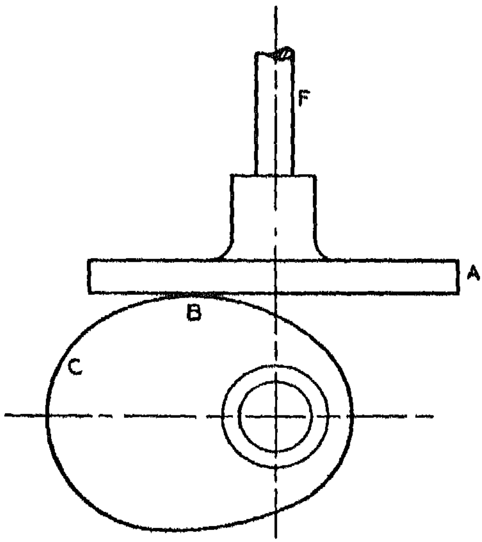


Fig. 168.

to the follower F and the rotating cam C causes a reciprocating motion of the follower. The line of motion of the follower may pass through the centre of the cam, or it may be displaced, as in Fig. 169. It will be noted that the point of contact B, Fig. 168, is not on the line of stroke. Taking a definite problem, let the angle of lift be 135°, the lift 1 in., the motion of the follower simple harmonic, the least radius $1\frac{3}{4}$ in., and the eccentricity of the follower $\frac{3}{4}$ in. In Fig. 169, let $OC = 1\frac{3}{4}$ in. represent the least radius, $OD = \frac{3}{4}$ in. the eccentricity. Assuming the follower to move in a vertical line, so that the flat plate is horizontal, a horizontal line drawn through C to meet BD, the line of stroke, at A will give the position of the plate for the commencement of lift.

AB is made equal to 1 in., the lift of the follower, and AB is divided for simple harmonic motion. Assuming that the line of stroke moves round the cam, the line of stroke is tangential to the circle of radius OD at the successive positions h, k, l,

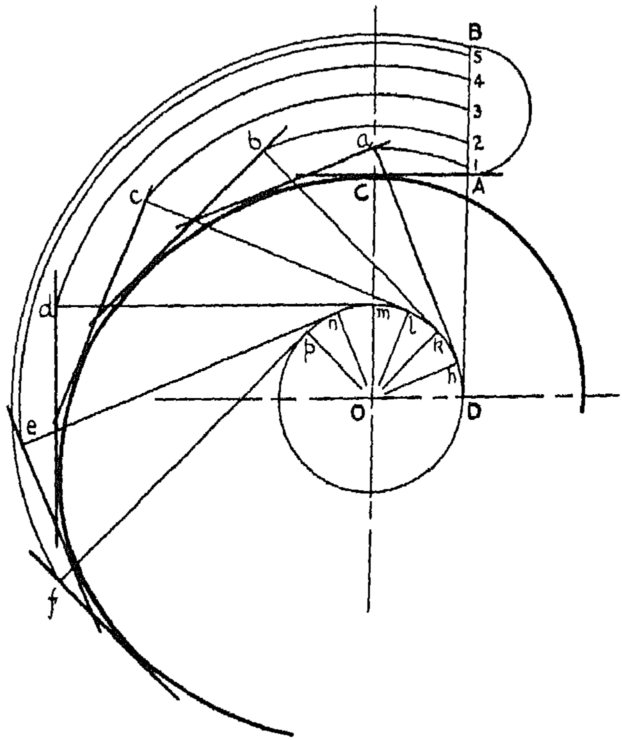


Fig. 169.

m, n, and p. The angle DOp is 135°. Swinging round from position 1, to cut ha at a, a line perpendicular to ha through a gives the position of the flat plate when the cam has moved through the angle DOh. The points b, c, d, e, and f are obtained in a similar manner, and lines drawn perpendicular to kb, lc, md, etc., give the respective positions of the flat plate. A smooth curve drawn tangential to these perpendicular lines gives the cam profile.

§ 202. **Oscillating Lever with Roller.** In Fig. 170, let O be the centre of a cam which is required to give an oscillating motion to a lever whose centre is at D. Let A and B represent the limits of motion of the centre of the roller which is fitted to the end of the lever. The circles at A and B represent the roller in the two extreme positions. Let the points 1, 2, 3, etc., divide the circular arc AB for simple

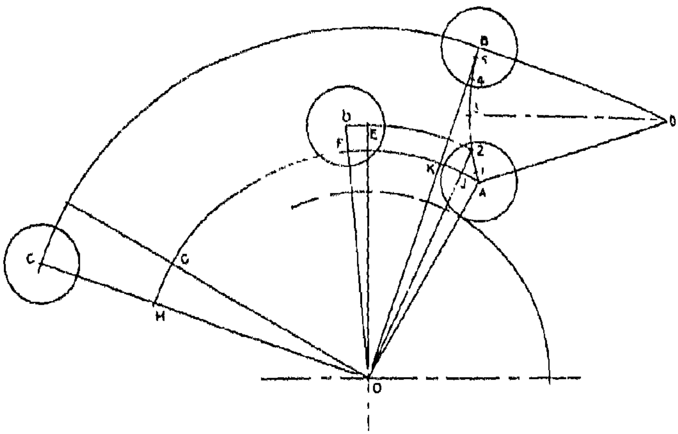
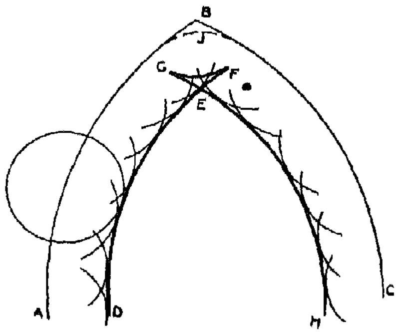


Fig. 170.

harmonic motion. Let the angle GOA be the angle of lift. As before, it is necessary to construct the pitch profile to represent the path of the centre of the roller relative to the cam. The oscillation of the lever about D causes an obliquity angle which varies according to the angle turned through by the cam. Let the angle AOE be the angle which the cam turns through while the centre of the roller moves to position 2. When E has moved to A, it has moved too far by the distance AJ or by the angle $AO2$. Making the angle EOb = angle $AO2$ and swinging round from 2, the point b is obtained on the pitch profile. The same result is obtained by making EF equal to AJ and producing OF to meet the circular arc $b2$ at b. The circle whose centre is at b represents the roller and the cam profile is tangential to this circle. Similarly when G has moved to A, it has moved too far by

the amount AK, and the corrected position of the roller centre is found by making $GH = AK$ and producing OH to C to cut the circular arc BC at C. The circle at C is the roller circle and the cam profile is tangential to this circle. The points b and C are points on the pitch profile, and by repeating the process for the points 1, 3, 4, and 5 the complete pitch profile may be obtained. The construction for these other points is omitted for the sake of clearness of diagram.



§ 203. **Interference of Cams.** The profile of a cam must be a continuous curve without loops, if the working face of the cam is an external face; a cam with a slot or groove in a side face, in which a pin or roller can fit, may, if necessary, have loops. When the pitch profile of a cam has been determined it sometimes happens that the envelope of the curve forms a loop as in Fig. 171, in which ABC is the pitch profile. The cam profile is $DEFGH$ and contains the loop EFG. The loop EFG cannot, of course, exist on the actual cam and must be cut away, leaving the cam profile as DEH. If the actual cam profile is DEH, the path moved over by the centre of the roller is AJC and does not conform with the original path ABC. The action in a case like this is due to *interference*. Interference is likely to occur when the radius of the roller is too large and when an angle such as ABC is too acute. Interference frequently occurs on cams where the follower is fitted

with a flat plate, and unless the lift is spread over a comparatively large angle, as in Fig. 169, it is difficult to arrange for the follower to have simple harmonic motion.

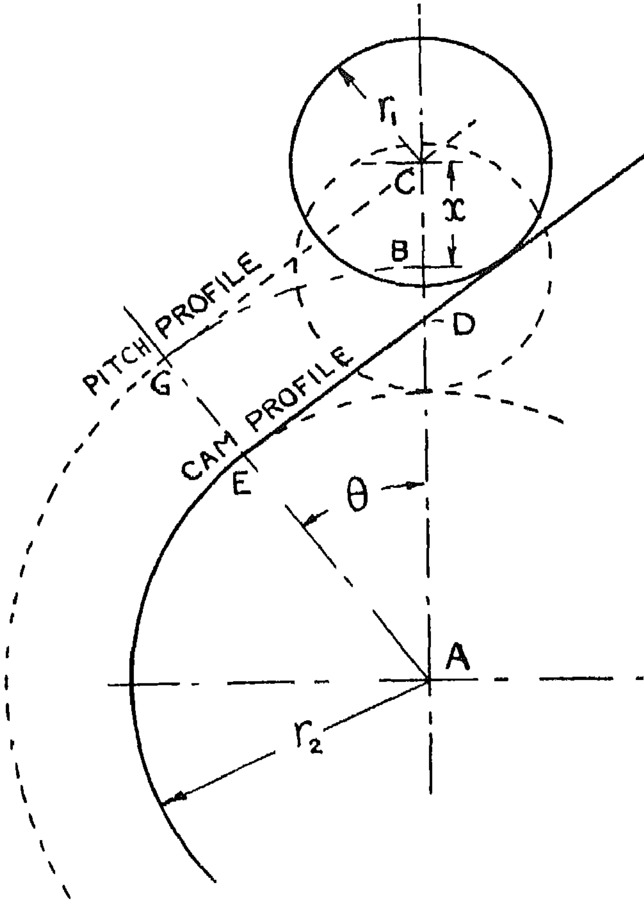


Fig. 172.

§ **204. Cam with Straight Flank.** Cams for operating the inlet and exhaust valves of internal combustion engines are frequently made with straight flanks and a nose consisting

of a circular arc. Expressions for displacement, velocity and acceleration of the follower can be found for such cams.

In Fig. 172 let A represent the centre of the cam axis, r_2 the least radius of the cam and r_1 the radius of the roller. The roller in its lowest position has its centre at B and when the cam has turned through an angle θ from the position in which lift is about to take place, the straight flank is represented by EF. The position of the centre of the roller is then C and the displacement of the roller from its bottom position is BC. The dotted line GC is the pitch profile.

Let x = displacement of roller,

$$\begin{aligned} x &= BC = AC-AB \\ &= AD+DC-AB \\ &= \frac{r_2}{\cos\theta}+\frac{r_1}{\cos\theta}-(r_1+r_2) \\ &= (r_2+r_1)\left(\frac{1}{\cos\theta}-1\right). \end{aligned}$$

Differentiating with respect to t to find velocity,

$$\begin{aligned} v = \frac{dx}{dt} &= (r_2+r_1)\frac{\sin\theta}{\cos^2\theta}\cdot\frac{d\theta}{dt} \\ &= \omega(r_2+r_1)\frac{\sin\theta}{\cos^2\theta}. \end{aligned}$$

Differentiating again to find acceleration,

$$\begin{aligned} a = \frac{dv}{dt} &= \omega(r_2+r_1)\left(\frac{\cos^2\theta\,.\cos\theta+2\cos\theta\sin\theta\,.\sin\theta}{\cos^4\theta}\right)\frac{d\theta}{dt}. \\ &= \omega^2(r_2+r_1)\frac{\cos^2\theta+2\sin^2\theta}{\cos^3\theta} \\ &= \omega^2(r_2+r_1)\left(\frac{2-\cos^2\theta}{\cos^3\theta}\right). \end{aligned}$$

§ **205. Circular Nosed Cam.** Expressions for displacement, velocity and acceleration for the follower as it moves over the circular nose of a cam can similarly be found. In Fig. 173 A is the centre of the cam shaft and r_2 the least radius of the cam. The centre of the circular nose is D and the

radius r_3. The centre of the roller when in contact with the circular nose is at C which lies on the pitch profile. In this case it is more convenient to measure displacement from the *top* position of the roller, and the centre of the roller when in

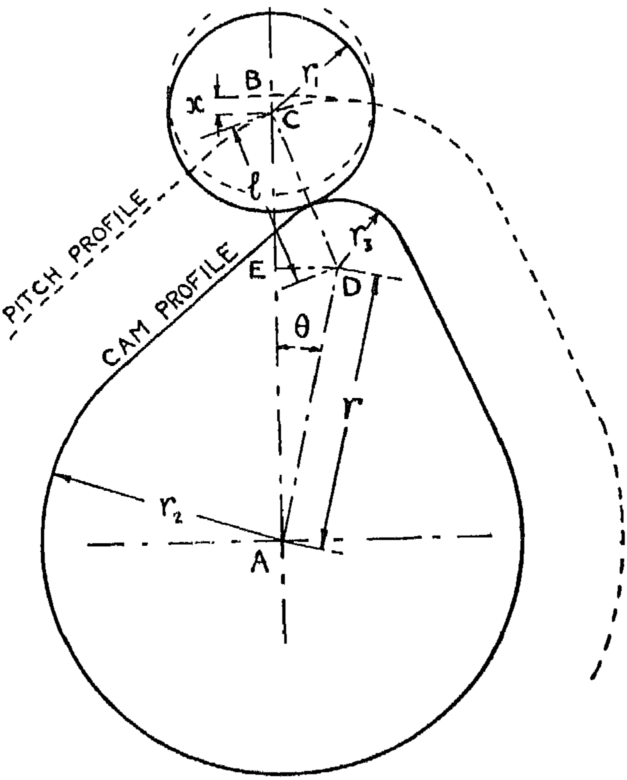


Fig. 173.

this position is B. The angle turned through by the cam measured from the position when the roller is at the apex is θ.

Let $AD = r$ and $DC = l = r_1+r_3$.

(Note that displacement is measured from top position.)

$$x = BC = AB-AC = AB-AE-EC$$
$$= l+r-r\cos\theta-\sqrt{(l^2-r^2\sin^2\theta)}.$$

Differentiating with respect to t to find velocity,

$$v = \frac{dx}{dt} = r\sin\theta . \frac{d\theta}{dt} + \frac{1}{2}\frac{2r^2\sin\theta\cos\theta}{(l^2-r^2\sin^2\theta)^{\frac{1}{2}}} . \frac{d\theta}{dt}$$

$$= \omega r\left(\sin\theta + \frac{r}{2}\frac{\sin 2\theta}{(l^2-r^2\sin^2\theta)^{\frac{1}{2}}}\right).$$

Differentiating again with respect to t to find acceleration,

$$a = \frac{dv}{dt} = \omega r\left(\cos\theta\frac{d\theta}{dt} + \frac{1}{2}\frac{(l^2-r^2\sin^2\theta)^{\frac{1}{2}} 2r\cos 2\theta + r\sin 2\theta\,\frac{2r^2\sin\theta\cos\theta}{2(l^2-r^2\sin^2\theta)^{\frac{1}{2}}}}{(l^2-r^2\sin^2\theta)} . \frac{d\theta}{dt}\right)$$

and this reduces to

$$a = \omega^2 r\left(\cos\theta + \frac{rl^2\cos 2\theta + r^3\sin^4\theta}{(l^2-r^2\sin^2\theta)^{\frac{3}{2}}}\right).$$

It will be noticed that these expressions for displacement, velocity and acceleration of the follower when in contact with a circular nose and measured from the *top* position of the follower are exactly the same as those given in Art. 67 for the motion of the piston in the slider crank chain.

§ **206. Convex Flanks.** Part of a cam with convex flanks is shown in Fig. 174 in which A is the centre of the cam shaft and r_2 the least radius of the cam. The centre of the roller when in its lowest position is B and when in contact with the convex flank the centre is C. The centre of the convex flank is at D and the radius r_3. The radius of the roller is r_1.

Let $l = CD = r_1 + r_3$,

$r = AD = r_3 - r_2$.

Displacement of centre of roller from *lowest* position is BC. The angle turned through by the cam measured from the position in which the roller is at its lowest is θ.

$$x = BC = EC - EA - AB$$

$$= \sqrt{(l^2-r^2\sin^2\theta)} - r\cos\theta - (l-r).$$

$$v = \frac{dx}{dt} = -\omega r\left(\frac{r\sin 2\theta}{2(l^2-r^2\sin^2\theta)^{\frac{1}{2}}} - \sin\theta\right).$$

$$a = \frac{dv}{dt} = -\omega^2 r\left(\frac{rl^2 \cos 2\theta + r^3 \sin^4\theta}{(l^2 - r^2 \sin^2\theta)^{\frac{3}{2}}} - \cos\theta\right.$$

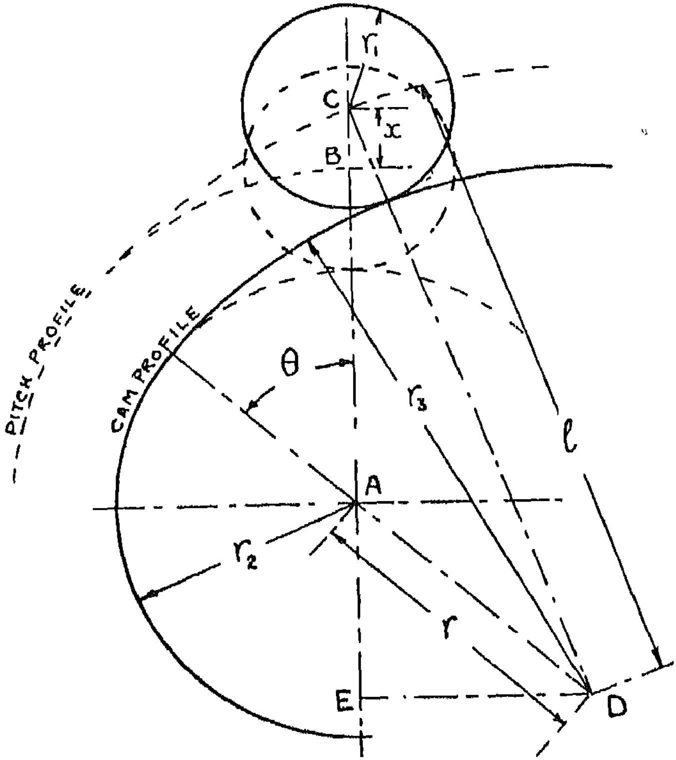


Fig. 174.

§ **207. Concave Flanks.** Part of a cam with concave flanks is shown in Fig. 175 in which A is the centre of the cam shaft and r_2 the least radius of the cam. The centre of the roller when its lowest position is B and when in contact with concave flank is C. The centre of the concave surface is at D and its radius is r_3. The radius of the roller is r_1.

Let $l = DC = r_3 - r_1$,

$r = AD = r_2 + r_3$,

θ = angle turned through by cam from the position in which the roller is at its lowest.

Displacement of centre of roller $= BC = x$,

$$x = BC = AE - CE - AB$$
$$= r\cos\theta - \sqrt{(l^2 - r^2\sin^2\theta)} - (r - l).$$
$$v = \frac{dx}{dt} = -\omega r\left(\sin\theta - \frac{r\sin 2\theta}{2(l^2 - r^2\sin^2\theta)^{\frac{1}{2}}}\right),$$
$$a = -\omega^2 r\left(\cos\theta \quad \frac{rl^2\cos 2\theta + r^3\sin^4\theta}{(l^2 - r^2\sin^2\theta)^{\frac{3}{2}}}\right).$$

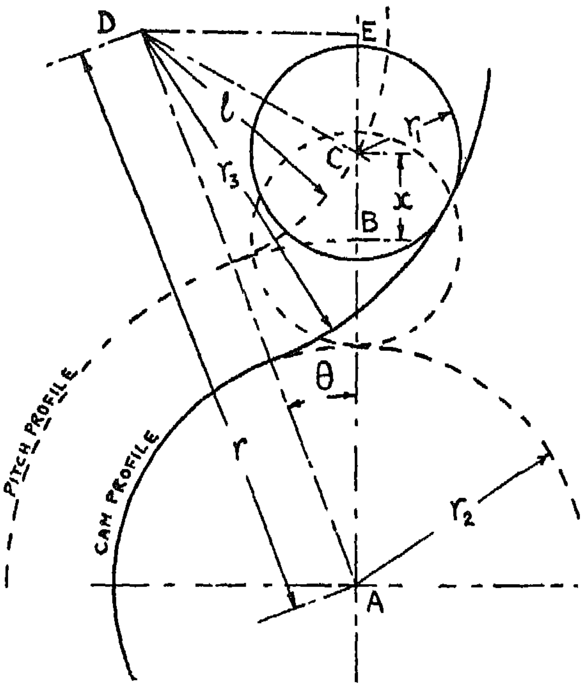


Fig. 175.

Again it will be noticed that these expressions for displacement, velocity, and acceleration for concave flanks are similar to the expression for displacement, velocity, and acceleration of the piston in the slider crank chain.

It is thus possible, though somewhat tedious, to calculate values and to plot curves of displacement, velocity, and acceleration against the cam angle for cams with either

straight, convex or concave flanks and with a circular nose. Since cams with straight flanks and circular noses are easier to manufacture than cams designed to give constant acceleration and constant deceleration or cams designed to give simple harmonic motion it will be found that in many practical cases cams with straight flanks and circular noses are used. If the radii of the roller, nose, and least radius are judiciously chosen it is possible to design a cam with straight flanks and circular nose to give motion very nearly equivalent to that of constant acceleration and deceleration.

§ **208. Straight Flanks with Circular Nose.** In Arts. 204 and 205 methods of calculating the displacement, velocity, and acceleration are given for straight flanks and circular nose respectively. To illustrate how these values approximate to those of constant acceleration and deceleration, the author has calculated values for the cam shown in Fig. 176 in which A is the centre of the cam shaft, C is the centre of the roller when in contact with the apex of the cam, C_1 is the centre of the roller when contact with the straight flank is just changing over to contact with the circular nose; the centre of the circular nose is D.

In Fig. 176 let

r_1 = radius of roller $= 2\frac{1}{2}$ in.

r_2 = least radius of cam $= 2$ in.

r_3 = radius of circular nose $= 0{\cdot}3$ in.

θ_1 = angle of lift $= 50°$

θ_2 = angle C_1AC = angle through which the cam moves from the position when contact with the circular nose begins to the position when contact is at apex.

Then $$r = AD = \frac{r_2 - r_3}{\cos\theta_1} = \frac{2-0{\cdot}3}{0{\cdot}6428} = 2{\cdot}6447 \text{ in.}$$

$$l = CD = r_1 + r_3 = 2{\cdot}5 + 0{\cdot}3 = 2{\cdot}8 \text{ in.}$$

$$\tan\theta_2 = \frac{C_1E}{EA} = \frac{l\sin\theta_1}{l\cos\theta_1 + r}$$

$$= \frac{2{\cdot}8\times0{\cdot}7660}{2{\cdot}8\times0{\cdot}6428+2{\cdot}6447} = 0{\cdot}4826.$$

Hence $$\theta_2 = 25°\,46'.$$

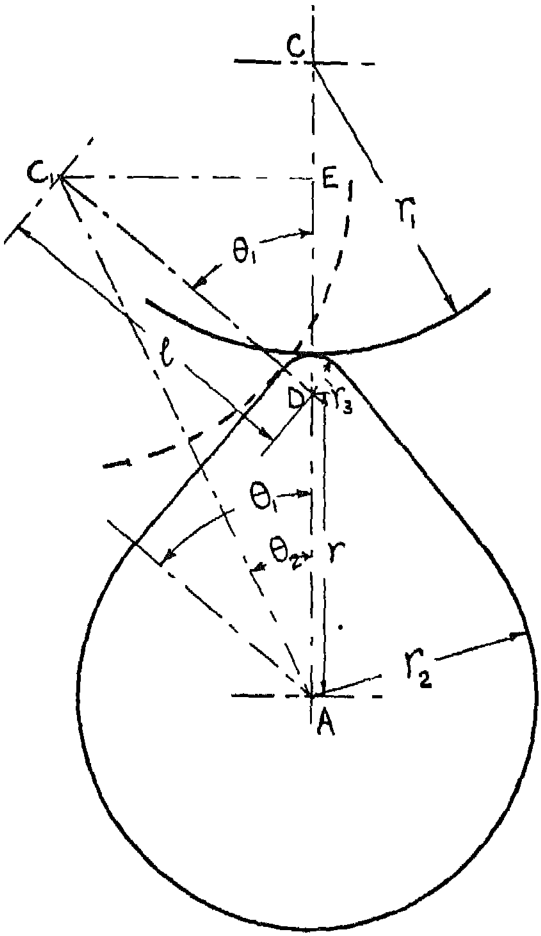


Fig. 176.

Height of tip of nose above least radius

$$= \text{lift of follower} = r+r_3-r_2$$
$$= 2{\cdot}6447+0{\cdot}3-2{\cdot}0$$
$$= 0{\cdot}9447 \text{ in.}$$

Values of displacement, velocity, and acceleration have been calculated for values of θ ranging from 0° to 25° in

intervals of 5° for contact with the straight flank, and for a cam speed of 1 radian per second.

These values are tabulated below.

Straight flank

θ *deg.*	*Displacement in.*	*Velocity in. per sec.*	*Acceleration in. per sec. per sec.*
0	0·0	0	4·50
5	0·0172	0·395	4·59
10	0·0694	0·805	4·85
15	0·159	1·248	5·33
20	0·289	1·742	6·06
25	0·465	2·315	7·12

Values also have been calculated for values of θ ranging from 0° to 25° in intervals of 5° for contact with the circular nose but in this case θ is measured from the apex of the cam (see Art. 205), and for a cam speed of 1 radian per second. These values are tabulated below.

Circular nose

(NOTE: θ is measured from apex of cam.)

θ *deg.*	*Displacement from apex in.*	*Displacement from lowest position in.*	*Velocity in. per sec.*	*Acceleration in. per sec. per sec.*
0	0·0	0·9447	0	5·15
5	0·0207	0·924	0·448	5·12
10	0·0777	0·867	0·892	5·05
15	0·1757	0·769	1·329	4·94
20	0·3087	0·636	1·754	4·78
25	0·4817	0·463	2·152	4·58

These values of displacement, velocity, and acceleration for both straight flank and circular nose are shown plotted in Fig. 177 against cam angle θ for lift only. The full lines show the curves for the cam shown in Fig. 176 and the dotted lines are the curves for constant acceleration and constant deceleration, each over half the lift. On comparing the two sets of curves it will be noticed that for the dimensions of the cam shown in Fig. 176 a very close approximation to constant acceleration and deceleration is obtained.

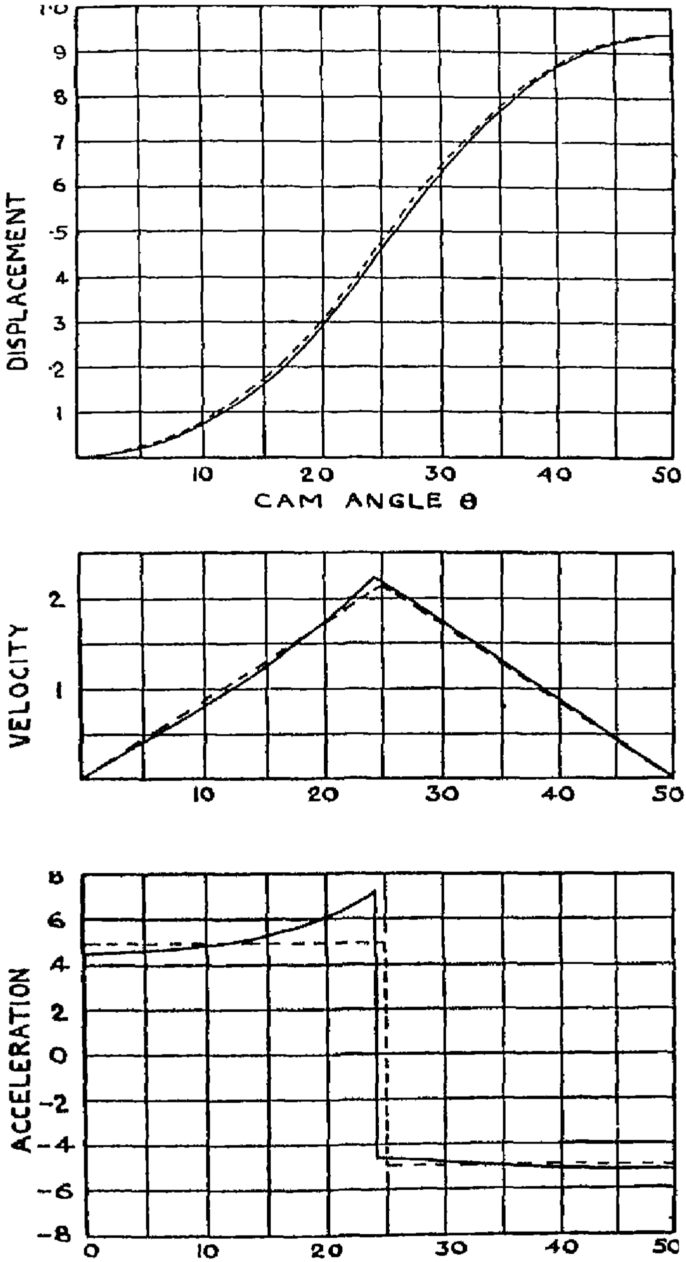

1·0
9
8
7
6
·5
4
3
·2
1
DISPLACEMENT
10
20
30
40
50
CAM ANGLE θ
2
1
VELOCITY
10
20
30
40
50
8
6
4
2
0
−2
−4
−6
−8
ACCELERATION
0
10
20
30
40
50

Fig. 177.

EXERCISES. XIII

1. Design a cam for lifting a valve to satisfy the following conditions: The cam shall raise the valve during one-sixth of a revolution, keep it fully open during one-quarter of a revolution, and close it during one-eighth of the revolution, the opening and closing motion being simple harmonic.

The lift of the valve is to be $1\frac{1}{2}$ in., the least radius of the cam $1\frac{3}{4}$ in., and the roller on the spindle is to be 1 in. diameter. The cam shaft has uniform motion and the axis of the spindle cuts the cam-shaft axis. [*I. Mech. E.*]

2. Design a cam which will give a lift of $1\frac{1}{2}$ in. to a rod carrying a $\frac{3}{4}$-in. diameter roller and the axis of which passes through the cam centre. The smallest radius of the cam is 2 in., the lifting of the rod is to be simple harmonic and is effected in a quarter revolution, and the dropping is to be as quick as possible at half revolution. [*I. Mech. E.*]

3. The profile of a cam is elliptical in shape and the major and minor axes of the ellipse are 5 and $3\frac{1}{2}$ in. respectively. The cam is keyed on a shaft so that the axis of the shaft passes through a point on the major axis of the ellipse and 1 in. from its centre. A tappet rod, the axis of which also passes through the axis of the shaft, is provided at its end with a roller 1 in. diameter, and the cam operates the tappet rod through this roller.

Using rectangular coordinates, draw a curve of tappet rod displacement upon a shaft angle base for a complete revolution of the shaft, and find the maximum velocity of the tappet rod if the shaft makes 200 revs. per min. [*I. Mech. E.*]

4. A rotating cam is required to impart simple harmonic motion to a follower during one half a revolution and to keep it at rest for the remaining half. Show how to construct the cam and how to find the acceleration of the follower during the time that it is being raised and lowered. [*Inst. C. E.*]

5. The magnitude of the maximum accelerating force for the valve of an internal-combustion engine is required. The lift in inches of the valve is measured for various angles of rotation of the crank-shaft and a curve plotted. Give, in general terms, the remaining data that must be known, and show clearly how the maximum accelerating force may be determined. [*Inst. C. E.*]

6. A circular cam is 4 in. in diameter and rotates about an axis 1 in. from the centre of the cam. A follower, fitted with a roller 1 in. in diameter, moves in a vertical line which is at a perpendicular distance of $1\frac{1}{4}$ in. from the axis of rotation of the cam. Draw, full size, a diagram showing the displacement of the follower from its lowest position for one revolution of the cam.

7. Design a cam which will lift a vertical spindle a distance of $1\frac{1}{2}$ in. while the cam makes $\frac{1}{3}$ rev., and allow the spindle to fall in $\frac{2}{3}$ rev., with

equal intervals of rest between these movements. The cam-shaft diameter is $1\frac{3}{4}$ in., the least metal is 1 in., and the diameter of the roller is 1 in. The spindle must rise and fall with simple harmonic

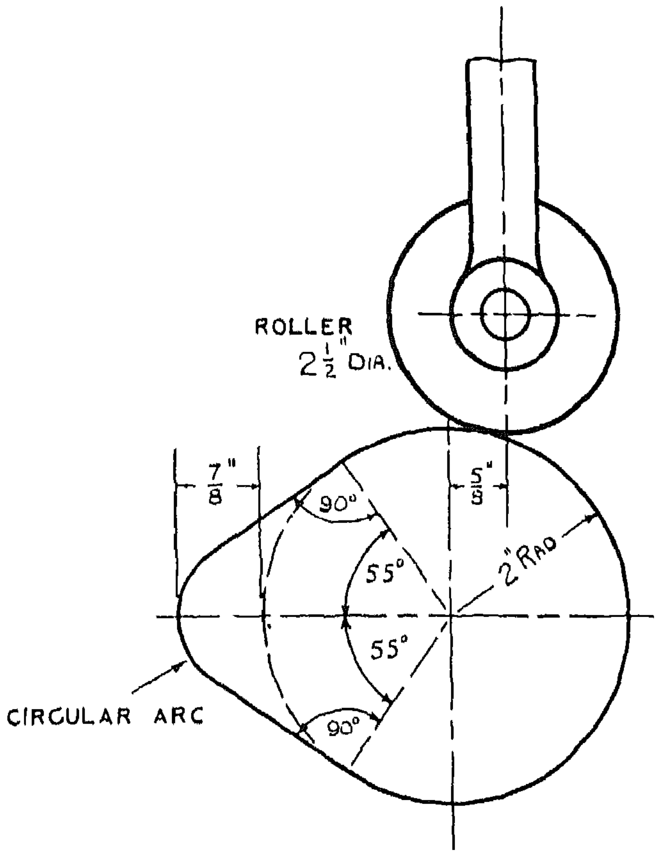


Fig. 178.

motion, and the line of stroke of the spindle passes through the centre of the cam shaft. The cam rotates with uniform speed.

8. OC is a straight line 3 in. long sloping downwards from O to C at an angle of 45° to the horizontal. C is the axis of a shaft upon which a cam is mounted. The edge of the cam works against the lower edge of a bar hinged at O. The lower edge of the bar is straight and passes through O. When the shaft rotates uniformly the bar receives

harmonic angular motion from the cam, the angular swing being 30°, and it make one complete oscillation for one revolution of the cam. The bottom edge of the bar is horizontal when in the middle of its swing.

Make a full-size drawing of the outline of the cam. [*Lond. B.Sc.*]

9. The axis of the valve spindle for a petrol engine is perpendicular to the axis of the cam shaft and distant $\frac{3}{4}$ in. from it at the nearest place. The driving face of the cam is an involute drawn to a circle of $1\frac{1}{2}$ in. diameter. The minimum radius of the cam is $\frac{5}{8}$ in., and the lift of the valve is $\frac{5}{8}$ in. The face of the cam works in contact with a roller $1\frac{1}{4}$ in. diameter.

Determine the maximum radius of the cam and the angle of cam action for the valve to receive its full lift. [*Lond. B.Sc.*]

10. A cam profile and tappet are shown in Fig. 178. Make a full-size drawing of the profile and roller, and obtain a diagram of the tappet's motion on a base-line along which $5\frac{1}{2}$ in. = 110°. The tappet operates a valve, and at the bottom of the movement there is $\frac{1}{16}$ in. clearance between the top of the tappet and the valve stem. Determine the angular movement of the cam during which the valve is open. [*Lond. B.Sc.*]

11. A cam, rotating in the direction shown in Fig. 179, has to move the centre of the tappet roller from A to B and back again while it rotates through an angle of 110° with uniform speed. In the diagram given the ordinates represent displacement of A along AB, drawn half-size, and the abscissae represent the corresponding angles turned through by the cam. Draw, full size, the outline of the cam, its minimum radius being $1\frac{1}{2}$ in. and the roller diameter $1\frac{1}{4}$ in.

[*Lond. B.Sc.*]

12. Design the profile of a cam which will give a lift of $1\frac{1}{2}$ in. to a follower carrying a roller 1 in. in diameter. The centre line of the follower passes through the centre of rotation of the cam, which rotates at 120 revs. per minute. The motion of the rod is to be simple harmonic and is raised its full amount in 0·1 second, remains at rest for 0·025 second and falls in 0·08 second. The cam rotates at constant speed and the least radius is 2 in. [*Lond. B.Sc.*]

13. A circular plate $3\frac{1}{4}$ in. diameter is used as a cam and rotates at 150 revs. per minute. The cam shaft centre is eccentric from the centre of the cam by $\frac{3}{4}$ in. The end of the follower carries a roller $1\frac{1}{4}$ in. in diameter and the line of stroke is 1 in. from the centre of the cam shaft.

Draw a diagram showing the displacement of the follower for a complete revolution of the cam. [*Lond. B.Sc.*]

14. A cam is to be keyed to a horizontal shaft 2 in. in diameter and is required to give simple harmonic motion to a spindle which can

move in the plane of the cam in guides inclined at 30 degrees to the vertical. The spindle carries a roller, $1\frac{1}{4}$ in. in diameter, in contact with the cam, and when the spindle is in its lowest position the centre of the roller is vertically above the centre of the cam. Taking the

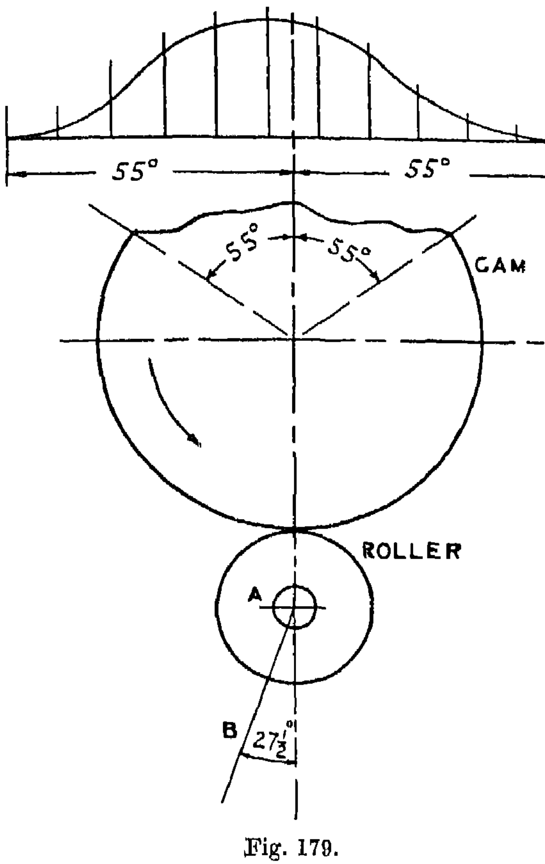


Fig. 179.

least thickness of metal to be $\frac{7}{8}$ in. draw the outline of the cam to give a movement of $1\frac{3}{4}$ in. to the spindle. [*Lond. B.Sc.*]

15. A valve has reciprocating motion and is to be operated by a cam which rotates at constant speed; the cam is kept in contact with a curved surface on the foot of the valve stem. The radius of curvature of this surface is 3 in. and the centre of curvature is on the centre line of the valve stem. The valve is to have simple harmonic motion with a lift of $1\frac{1}{2}$ in. and the line of motion passes through the centre of

rotation of the cam. Taking the least radius of the cam to be 2 in. construct the cam profile to satisfy the following conditions:

Angle turned through by cam	*Motion of valve*
70 degrees	lifting
20 ,,	full open
50 ,,	falling
220 ,,	closed

[*Lond. B.Sc.*]

16. Draw, full size, the outline of a cam of least radius 2 in. to give reciprocating motion to a follower fitted with a roller 1 in. in diameter. The angle of lift is 60° and the angle of fall is 60° with no interval between. The follower has a lift of $1\frac{1}{4}$ in. and moves with simple

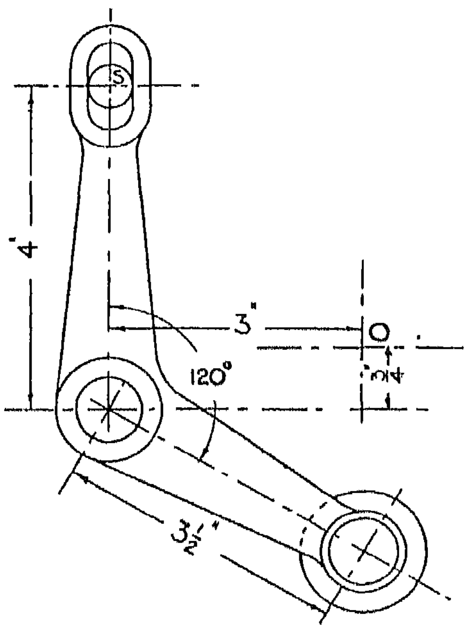


Fig. 180.

harmonic motion in a vertical path, the line of action being displaced by $\frac{3}{4}$ in. to the right of the cam centre when viewed from front elevation. [*Lond. B.Sc.*]

17. A cam is required to give motion to a follower fitted with a roller 2 in. in diameter which is in contact with the cam. The lift of the follower is $1\frac{1}{4}$ in. and it has uniform acceleration for $\frac{1}{2}$ in. while

the cam turns through 60°, uniform velocity for $\frac{1}{2}$ in. while the cam turns through 30° and uniform retardation for the remainder of the lift while the cam turns through 30°. The follower falls immediately with simple harmonic motion while the cam turns through 90° and the period of rest is 150° of cam angle.

Construct a lift and fall diagram on a cam angle base, and draw the outline of the cam whose least radius is $1\frac{1}{2}$ in. when the line of motion of the follower passes through the centre of the cam axis.

[*Lond. B.Sc.*]

18. Fig. 180 shows a bell-crank lever which is required to move the spindle *S* horizontally. This spindle has a total movement of $1\frac{1}{2}$ in. and when the longer arm of the bell-crank lever is vertical the spindle is in its mid position. The shorter arm carries a roller $1\frac{1}{2}$ in. diameter which is actuated by a cam whose centre of rotation is at *O*. Assuming that the cam rotates at constant speed, draw, full size, the outline of the cam to move the spindle with simple harmonic motion when the cam rotates in a clockwise direction. [*Lond. B.Sc.*]

APPENDIX

ACCELERATION DIAGRAM FOR QUICK-RETURN MOTION

As already explained in Art. 58, the drawing of an acceleration diagram for a quick-return motion, or for any mechanism in which sliding occurs on a rotating link, is more complex than for simple link mechanisms and before such a diagram can be drawn it is necessary to consider the accelerations arising from sliding accompanied by rotation.

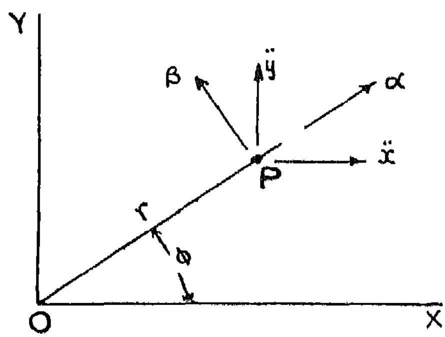


Fig. 181.

The acceleration of a point P, Fig. 181, having plane motion can be expressed in terms of the rectangular components $\ddot{x}$ parallel to the axis OX and $\ddot{y}$ parallel to the axis OY. If the point P is rotating around a centre such as O it is frequently more convenient to express component accelerations in terms of polar co-ordinates; these may be represented by α along OP and β perpendicular to OP; α is thus a radial acceleration and β a tangential acceleration. Considering the general case when OP is variable,

let $\quad OP = r$

$\qquad \phi =$ inclination of OP with OX

then $\quad x = r \cos \phi$

and $\quad y = r \sin \phi$

where x and y are the rectangular co-ordinates for the displacement of P at any instant.

Differentiating these with respect to time,

$$x = -r\phi \sin \phi + \dot{r} \cos \phi$$

$$y = r\phi \cos \phi + r \sin \phi$$

Differentiating again with respect to time,

$$x = -r\dot{\phi}^2 \cos \phi - \dot{r}\phi \sin \phi - r\ddot{\phi} \sin \phi + \dot{r} \cos \phi - r\dot{\phi} \sin \phi$$

$$y = -r\dot{\phi}^2 \sin \phi + \dot{r}\phi \cos \phi + r\phi \cos \phi + r \sin \phi + r\phi \cos \phi$$

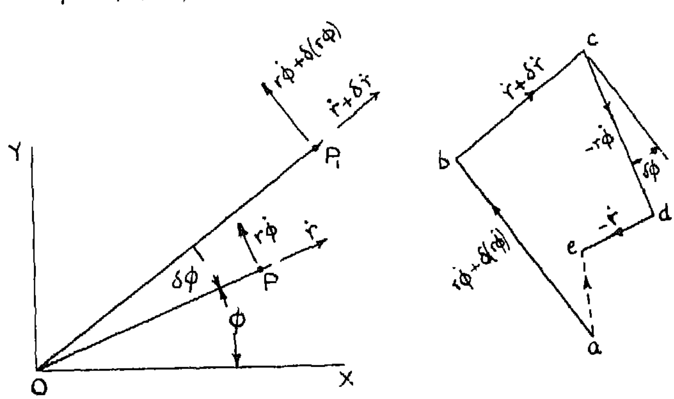
now $\alpha = x \cos \phi + \ddot{y} \sin \phi$

and $\beta = y \cos \phi - x \sin \phi$

Substituting values of x and $\dot{y}$ in these expressions for α and β, we get,

$$\alpha = r - r\phi^2$$

and $\beta = r\phi + 2r\dot{\phi}$



(a) Fig. 182 (b)

An alternative treatment of this problem is to consider the vectors representing the initial and final velocities of P when OP has turned through an angle $\delta\phi$ in time δt. The change of velocity is the final velocity minus the initial velocity and the acceleration is this change of velocity divided by δt.

In Fig. 182(a)

let ϕ = inclination of OP to axis OX,

r = distance OP

Then $\dot{r}$ = initial velocity of P along OP, i.e radially,

and $r\phi$ = initial velocity of P tangential to OP.

After a short interval of time the link OP has moved through the angle $\delta\phi$ in δt seconds and P has moved to P_1.

The final velocity at P_1 is given by

$r + \delta r$ radially along OP_1

and $r\phi + \delta(r\phi)$ tangential to OP_1.

The vectors representing the initial and final velocities and the change of velocity are shown in Fig. 182(*b*) in which the final velocity is represented by the vectors *ab* and *bc*; the vectors *cd* and *de* represent *minus* the initial tangential and radial velocities of P respectively and *ae* is the sum of *ab*, *bc*, *cd* and *de* and thus represents the change of velocity as P has moved from P to P_1.

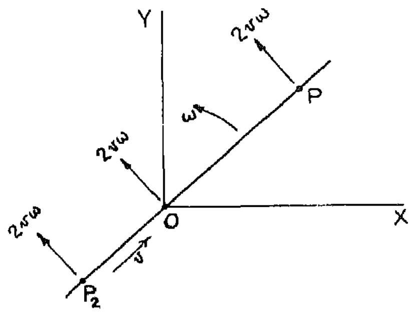


Fig. 183.

Tangential change of velocity = component of *ae* in the direction of *ab*

$$= r\dot{\phi} + \delta(r\dot{\phi}) - r\dot{\phi}\cos\delta\phi + \dot{r}\sin\delta\phi$$

As $\delta\phi$ approaches zero, $\cos\delta\phi$ approaches unity and $\sin\delta\phi$ approaches $\delta\phi$.

$$\text{Hence tangential acceleration} = \frac{r\dot{\phi} + \delta(r\dot{\phi}) - r\dot{\phi} + \dot{r}\delta\phi}{\delta t}$$

$$= \dot{r}\dot{\phi} + r\ddot{\phi} + \dot{r}\dot{\phi}$$

$$= r\ddot{\phi} + 2\dot{r}\dot{\phi}$$

$$\text{Radial change of velocity} = \dot{r} + \delta\dot{r} - r\dot{\phi}\sin\delta\phi - \dot{r}\cos\delta\phi$$

$$\text{and radial acceleration} = \frac{\dot{r} + \delta\dot{r} - r\dot{\phi}\,\delta\phi - \dot{r}}{\delta t}$$

$$= \ddot{r} - r\dot{\phi}^2$$

The complete acceleration of P is thus represented by four components of acceleration, $r\ddot{\phi} + 2\dot{r}\dot{\phi}$ tangentially and $\ddot{r} - r\dot{\phi}^2$ radially. If r is constant these component accelerations reduce to $r\ddot{\phi}$ tangentially and $-r\dot{\phi}^2$ radially—the accelerations which have previously been discussed (Art. 51) for a point rotating at constant radius. When r is variable two additional components of acceleration are thus induced, $2\dot{r}\dot{\phi}$ tangentially and $\ddot{r}$ radially; $2\dot{r}\dot{\phi}$ is accounted for by the change in the tangential velocity of

P as it moves along OP and by the change in direction of the tangential velocity of P; $\dot{r}$ is the acceleration with which P slides along OP. The tangential component of acceleration $2\dot{r}\dot{\phi}$ is known as the Coriolis component.

It will be observed that the direction of the Coriolis component $2r\phi$ is in the direction in which P tends to move tangentially and the magnitude of $2\dot{r}\phi$ depends only upon $\dot{r}$ and ϕ and does not depend upon the position of P along OP. It has been assumed that r and ϕ are positive quantities and that the increments δr and $\delta\phi$ are positive; if δr is negative, i.e. P is moving towards O the direction of the Coriolis component is reversed and is in a direction opposite to that in which P would tend to move. In more general terms if r is replaced by v the velocity of sliding and ϕ by ω the angular velocity of OP the Coriolis component then becomes $2v\omega$. Thus in Fig. 183 if a body is moving from P_2 to P with velocity v and P_2OP is rotating in a counter-clockwise direction with angular velocity ω, the Coriolis component is constant in magnitude and direction for all positions of the body and its value is $2v\omega$. The convention for the sign of the Coriolis component thus becomes: if v is in a direction radially outwards as at P, the direction of the Coriolis component is in the direction which the body tends to move due to rotation; if v is in a direction radially inwards as at P_2, the direction is opposite to that which the body would tend to move due to rotation.

To assist in the identification of vectors representing the different components of acceleration it is convenient to adopt a system of notation similar to that explained in Art. 52, and to form vector equations representing relative accelerations. Thus if P is a moving point on a link OA which is rotating about a centre O we may consider a point Q on the link which is at a constant distance from O and which is coincident with P at the instant considered. In Fig. 184(a) let P be moving along OA and Q be a point on the link OA immediately beneath P. The vector equation connecting the accelerations of P, Q and O can now be formed.

Acc. of P to O=acc. of P to Q+acc. of Q to O. The acceleration of P to O has the components $r-r\phi^2$ radially and $r\dot{\phi}+2r\phi$ tangentially. The acceleration of Q to O has the components $-r\phi^2$ radially and $r\dot{\phi}$ tangentially; thus the acceleration of P to Q is r radially and $2\dot{r}\phi$ tangentially.

The vector equation can thus be expanded

$$\text{acc. of } P \text{ to } O = \underset{(\ddot{r}+2\dot{r}\dot{\phi})}{\text{acc. of } P \text{ to } Q} + \underset{(r\ddot{\phi}-r\dot{\phi}^2)}{\text{acc. of } Q \text{ to } O}$$

In any given problem, assuming the accelerations on the right hand side of the equation are known, the vector diagram for accelerations can be drawn in accordance with the method of notation previously adopted. In Fig. 184(b) o_1q_0 is drawn parallel to OQ and of length proportional to $r\dot{\phi}^2$, the direction being from Q to O; q_0q_1 is drawn perpendicular to OQ and of length proportional to $r\ddot{\phi}$ and o_1q_1, if joined, represents the acceleration of Q relative to O; q_1q_p is drawn parallel to the path P relative to Q to represent the acceleration of sliding $\ddot{r}$, and q_pp_1 is drawn perpendicular to OP to represent the Coriolis component of acceleration $2\dot{r}\dot{\phi}$; q_1p_1, if joined, represents the acceleration of P relative to Q. The sum of these vectors is o_1p_1 which represents the acceleration of P to O.

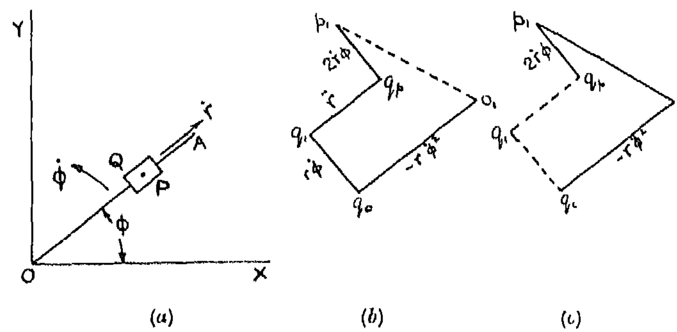


Fig. 184.

In many actual cases, however, the motion of P is controlled by some other link and the angular acceleration of OQ, $\ddot{\phi}$ and the acceleration of sliding $\ddot{r}$ may be unknown. If the motion of P is controlled by some other link the acceleration of P relative to O will be known and consequently o_1p_1 (Fig. 184(c)) can be drawn. The vectors $o_1q_0(-r\dot{\phi}^2)$ and q_pp_1 $(2\dot{r}\dot{\phi})$ are also known and the intersection of q_pq_1 and q_0q_1 locates the point q_1 to give the values of $q_1q_p(\ddot{r})$ and $q_0q_1(r\ddot{\phi})$. Values of $\dot{r}$ and $\dot{\phi}$ are obtained from the velocity diagram and hence the Coriolis component $2\dot{r}\dot{\phi}$ and the radial acceleration $-r\dot{\phi}^2$ can be readily calculated.

To illustrate the construction of an acceleration diagram involving the use of the Coriolis component of acceleration the Whitworth quick-return motion, shown diagrammatically in Fig. 185, will be solved. In this diagram A_1 and A_2 are fixed centres and the crank A_1P rotates at a constant speed. The lever CA_2D rotates about A_2 with variable angular velocity and Q is

a point on the lever coincident with P for the configuration considered. The following dimensions have been used:

$A_1A_2=2$ in.; $A_1P=4$ in.; $A_2D=4$ in.; $DE=16$ in.; angle $BA_1P=60°$ and the crank A_1P is rotating at a constant of 60 revolutions per minute in a clockwise direction.

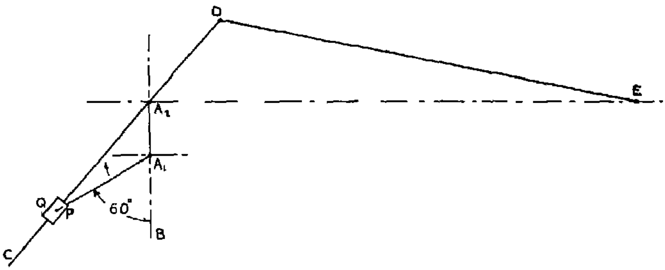


Fig. 185.

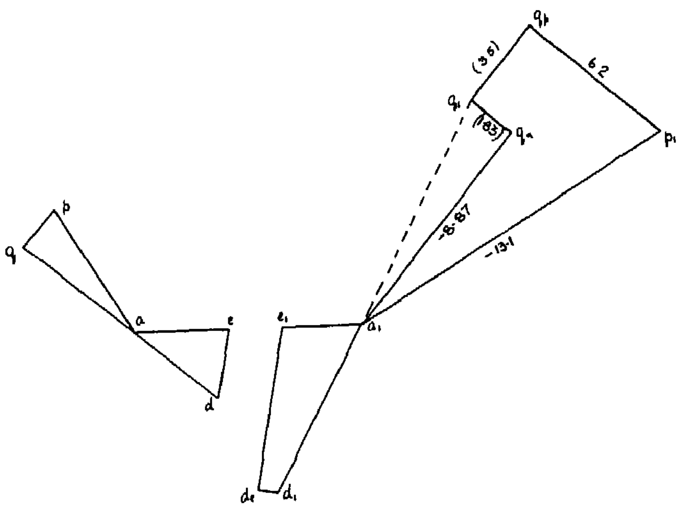


Fig. 186

From this data velocity and acceleration diagrams will be drawn to determine

(*a*) the velocity and acceleration of P along CA_2D;
(*b*) the angular velocity and angular acceleration of CA_2D;
(*c*) the angular velocity and angular acceleration of DE.
(*d*) the linear velocity and acceleration of the tool box at E.

Velocity diagram. The angular velocity of CA_2D can be determined when the velocity of any point such as Q on CA_2D is known; hence the first step is the determination of the velocity of Q.

vel. of P relative to A=vel. of P relative to Q+vel. of Q relative to A.

$$\text{vel. of } P \text{ relative to } A=\frac{60}{60}\times 2\pi\times\frac{4}{12}=2{\cdot}09 \text{ ft. per sec.}$$

In Fig. 186, ap is drawn perpendicular to A_1P and proportional to 2·09; aq is drawn perpendicular to CA_2D, since Q is a point on CA_2D and has, therefore, a velocity in a direction perpendicular to CA_2D; pq is drawn parallel to CA_2D to represent the velocity of P relative to Q. The intersection of aq and pq locates q. The velocity of P relative to Q, which is the velocity of P sliding along CA_2D, scales 0·69 ft. per sec., and aq represents the velocity of Q relative to A_2. The length A_2Q (Fig. 185) scales 5·3 in. and aq (Fig. 186) scales 1·98 ft. per sec.

$$\text{Hence the angular velocity of } A_2Q \text{ (or } CA_2D)=\frac{1{\cdot}98}{\frac{5{\cdot}3}{12}}=4{\cdot}49 \text{ rads. per sec.}$$

The velocity of D, being a point on CA_2D, is found by producing qa to d such that

$$\frac{aq}{ad}=\frac{A_2Q}{A_2D}$$

The vector equation connecting D, E and A is now formed (A being regarded as a fixed point with no velocity).

Vel. of E relative to A=vel. of E relative to D+vel. of D relative to A.

The velocity of D relative to A is known and is represented by ad; de is drawn perpendicular to DE and ae is drawn horizontally since E is constrained to move horizontally by guides. The intersection of ae and de locates the point e. From the diagram ae scales 1·32 ft. per sec. and this is the velocity of sliding of E; de scales 1·01 and hence the angular velocity of

$$DE=\frac{1{\cdot}01}{\frac{16}{12}}=0{\cdot}758 \text{ rads. per sec.}$$

Acceleration diagram. Vector equations, similar to those formed for the construction of the velocity diagram, can now be formed to facilitate the construction of the acceleration diagram.

The first vector equation thus becomes

acc. of P relative to A=acc. of P. relative to Q+acc. of Q relative to A.

Each of these accelerations has two components:

P relative to A—radial and tangential;
P relative to Q—sliding and Coriolis;
Q relative to A—radial and tangential.

Of these six component accelerations four are known, i.e.

radial acc. of P to $A = -\dfrac{(2\cdot09)^2}{\frac{4}{12}} = -13\cdot1$ ft. per sec. per sec.;

tangential acc. of P to $A=O$, since A_1P is rotating at constant angular velocity;

Coriolis acc. of P to $Q=2\times0\cdot69\times4\cdot49=6\cdot2$ ft. per sec. per sec.

radial acc. of Q to $A = -\dfrac{(1\cdot98)^2}{\frac{5\cdot3}{12}} = -8\cdot87$ ft. per sec. per sec.

In Fig. 186, a_1p_1 is drawn parallel to PA_1 and in a direction from P towards A_1 and proportional to $13\cdot1$; a_1q_a is drawn parallel to QA_2 and in a direction from Q towards A_2 and proportional to $8\cdot87$. The Coriolis component of acceleration q_pp_1 is drawn perpendicular to A_2Q and since P is sliding inwards towards A_2, the direction is opposite to that in which Q tends to move tangentially; q_pp_1 is thus made proportional to $6\cdot2$ and the intersection of q_pq_1 drawn parallel to A_2Q and q_aq_1 perpendicular toA_2Q locates q_1.

The acceleration of sliding of P relative to Q is represented by q_1q_p and this scales $3\cdot5$ ft. per sec. per sec. The tangential acceleration of Q relative to A_2 is represented by q_aq_1 and this scales $1\cdot83$ ft. per sec. per sec., hence the angular acceleration of A_2Q (i.e. CA_2D)$=\dfrac{1\cdot83}{\frac{5\cdot3}{12}} = 4\cdot14$ rad. per sec. per sec. and is in the same direction as the angular velocity.

The acceleration of D is found by joining q_1 and a_1 and producing q_1a_1 to d_1 such that

$$\frac{q_1a_1}{a_1d_1} = \frac{QA_2}{A_2D}$$

The second vector equation can now be formed, viz.,

acc. of E relative to A = acc. of E relative to D+acc. of D relative to A.

Each of these accelerations has two components:

E relative to A—radial and tangential (or sliding);

E relative to D—radial and tangential;

D relative to A—radial and tangential.

Of these six component accelerations four are known, i.e.

radial acc. of E to $A = O$ since E is moving a straight path;

radial acc. of E to $D = -\dfrac{(1\cdot01)^2}{\frac{16}{12}} = -0\cdot765$ ft. per sec. per sec.

The radial and tangential acc. of D to A is represented by a_1d_1 and is now known.

In Fig. 186, d_1de is drawn parallel to DE, proportional to 0·765 and in a direction from E to D; d_ee_1 is drawn perpendicular to DE and a_1e_1 parallel to the line of stroke of E. The intersection of d_1de and a_1e_1 locates e_1 and a_1e_1 scales 2·97 ft. per sec. per sec. which is the acceleration of the tool box at E. The angular acceleration of $DE = \dfrac{d_ee_1}{DE} = \dfrac{5\cdot95}{\frac{16}{12}} = 4\cdot47$ rads. per sec. per sec.

ANSWERS

I (pp. 17–19)

1. 0·489 ft. per sec. per sec.
2. 0·367 ft. per sec. per sec.; 0·113 rad. per sec. per sec.
3. 39·6 ft. per sec.; 82·7° with radius.
4. 32·5° E. of N.; 2·39 hours. 5. 19·2 knots; 4·7 naut. miles.
6. 2 ft. per sec. per sec.; 3 ft. per sec.; 990 ft.
7. 0·489 ft. per sec. per sec.
8. 2·4 ft. per sec. per sec.; 1·4 ft. per sec.
9. 13·5° S. of W.; 88·5 miles per hour.
10. 59·7 m.p.h.; 23° E. of N. 11. 1·494 knots; 6° 4′ S. of E.
12. 49° N. of W. 13. 4,080 ft. 14. 0·326 ft. per sec. per sec.
15. 182 ft. 16. 69·8° S. of E.; 1·193 hours; 23·86 sea miles.
17. $2\frac{1}{3}$ m.p.h.; 14·4 secs.
18. 313 ft. per sec.; 19·44 secs.; 1,502 ft.
19. 14·95 m.p.h.; 3·38 miles; 37·8 min.
20. 36 ft. per sec. 21. 8·75 revs.
22. (*a*) 19·63 min.; (*b*) 4·41 naut. miles; (*c*) 3·68 naut. miles; (*d*) 34·7 min.
23. 408 ft. per sec. per sec.
24. 161·4 ft. per sec.; 0; 2,000 ft. per sec. per sec.; 2,000 ft. per sec. per sec.
25. 28 ft. per sec. per sec.

II (pp. 36–40)

1. (*a*) 845 lb.; (*b*) 1,000 lb.; (*c*) 1,155 lb.
2. 4·02 ft. per sec. per sec.; 1,125 lb.
3. 18·12 secs.; 407 ft.; 97·1 ft.-lb.; 6,214 ft.-lb.
4. 254 lb. 5. 11·2 tons-ft. 6. 11·2 ft. per sec.; 18·25 tons.
7. 46·5 secs. 8. 50 secs. 9. £1. 1*s*. 3*d*.
10. 102 ft.-tons; 7·5 H.P. 11. 61 m.p.h.
12. 7·33 tons; 6·21 min.; 197 H.P.-hours.
13. 4,290 lb.; 458 H.P.; (*a*) 0·1069 ft. per sec. per sec.; (*b*) 0·187 ft. per sec. per sec.; (*c*) 9,205 ft.
14. 201·5 H.P.; 106·4 H.P.
15. 3·52 secs.; 20·39 secs.; 1·09 secs.; 35·2 ft. per sec.
16. 3,445 ft.-tons; 191 ft.-tons per sec. 17. 1·455 lb.
18. 3,740 lb. 19. (*a*) 212,000; (*b*) 7,350 lb.-ft.
20. 77·5 ft. per sec. 21. 1,876 lb.-ft.; 6·8 H.P.
22. 5,520 ft.-tons; 3·9 tons-ft.; 0·813 ton.
23. 10·15 tons-ft. 24. 21·1 secs.; 105·5 secs.
25. (*a*) 3·14 ft. per sec. per sec.; (*b*) 8·74 secs.; (*c*) 6,150 lb.
26. 8 mins. 16 secs.
27. (1) 307 secs.; 10·43 m.p.h.; (2) 235 secs.; 36·8 m.p.h.

III (pp. 61–68)

1. 25·6 ft. per sec.; 25·1 ft. per sec.
2. 262; 325; 23·3. 4. 1·205 ft. per sec.
5. 0·675 rad. per sec.; 1·73 ft. from C; 2·0 ft. per sec.
6. 4 in.; 17·0 ft. per sec. 7. 0·514; 37° with AE.
8. 4·79 ft. per sec.; 2·12 ft. per sec. 9. 13·94 in.; 9·06 ft. per sec.
10. 69°; 5·5 rads. per sec.; 0·74 rad. per sec.
11. A 28·0; B 41·9; C 30·7; D 17·0 ft. per min. 12. 92 lb.
13. 7·17 ft. per sec.; $4\frac{1}{2}°$ to line of centres. 14. 3·14; 1,130 lb.
15. (a) 8·71 ft. per sec.; (b) 1·875 rads. per sec.; (c) 1·543 ft. per sec.
16. 474; 43·5 ft. per min. 17. 7·81 rads. per sec.
18. 2·22 ft. per sec.
19. (a) 2·54 in.; (b) 2·53 in.; (c) 16·6 and 22·6 ft. per min.; (d) 20·4 ft. per min.
20. 25·8 rads. per sec.; 17·3 ft. per sec. 21. 76·4 ft. per sec.
22. 21·2 rads. per sec.; 73·3 in. per sec.
23. 35·8 ft. per sec.; 35·5 ft. per sec.
24. 29·5 in. per sec.; 5·65 rads. per sec.

IV (pp. 84–88)

1. 31 ft. per sec.; 149 ft. per sec. per sec.
2. 19·9 ft. per sec.; 290 ft. per sec. per sec.
3. 0·636 rad. per sec.; 1·055 rads. per sec. per sec.
4. 92·5 ft. per sec. per sec.; 68·0 ft. per sec. per sec.
6. 84·5 ft. per sec. per sec.; 72·6 rads. per sec. per sec.
7. 18·2 in.; 10·74 ft. per sec.; 89 ft. per sec. per sec.
8. 9·3 ft. per sec.; 175 ft. per sec. per sec.
9. 6·44 ft. per sec.; 103 ft. per sec. per sec.
10. 46·5 ft. per sec. per sec.
11. 11·5 and 15·5 rads. per sec. per sec.
12. 1·28 ft. per sec.; 0·90 ft. per sec.; 13 ft. per sec. per sec.
13. 2 to 1; 0·85 ft. per sec.; 1·0 ft. per sec. per sec.
14. 2·23 ft. per sec.; 7·5 ft. per sec. per sec.
15. 3·92 ft. per sec.; 16·26 ft. per sec. per sec.

V (pp. 101–103)

1. 128·7 ft. per sec. per sec.; 7·49 ft. per sec.; 103 ft. per sec. per sec.
4. 518 ft. per sec. per sec.; 44·5 per cent. of stroke.
9. 697 ft. per sec. per sec. 10. 155 ft. per sec. per sec.
11. −127·8 ft. per sec. per sec.
12. 326; 318; 172; −95; −163; −148; −148 ft. per sec. per sec.; 43 per cent.; 72°
13. 285 ft. per sec. per sec.
14. 924 and 850 ft. per sec. per sec. 15. 311 and 6·65 ft.-lb.

VI (pp. 134–136)

7. 7 in.; 21 in.; 14 and 42 teeth.
11. $\theta_1 = \theta_2 = 30°$; dia. pitch $= 2$; 7 in. and 21 in.; 14 and 42 teeth.

12. $4\frac{1}{2}$ and $1\frac{1}{2}$ ft. 13. (*a*) 13·925 in.; (*b*) 20 in.; (*c*) 21 in.
14. 38 and 133; 4 ft.—$0\frac{6}{7}$ in.; 3·54.
15. 30 and 99 teeth; 25·8 in.; 50 and 165 teeth; $s = 4\frac{1}{4}$.
17. 0·11 in. on pinion; 0·12 in. on wheel.
18. (*a*) 21 and 42 teeth; (*b*) 50° and 40°; (*c*) 0·777 in. and 0·653 in.; (*d*) 5·200 in. and 8·728 in.; 6·964 in.
19. 18 and 72 teeth; $p = 1\frac{3}{4}$.
20. 0·13 in. on pinion; 0·175 in. on wheel.
21. A, 26; C, 40, $s = 4$; B, 39; D, 93, $s = 8$.
22. A, 20; C, 32; B, 14; D, 38; $6\frac{1}{2}$ in.

VII (pp. 170–180)

1. 400 r.p.m. counter-clockwise.
3. 62 r.p.m.; $29\frac{11}{21}$ r.p.m. 4. 46·5.
5. 20 on lathe spindle; 35 and 30 compound; 60 on leadscrew.
6. $+\frac{3}{7}$ revs.; -3 revs. when B makes $+1$ rev.
7. (*a*) $1 - \dfrac{A.C}{B.D}$; (*b*) $\dfrac{A.C}{B.D}$; A and C $\frac{40}{7}$ in.; B and D $\frac{2}{7}$ in.
8. $\frac{1}{18}$. 9. (*a*) 30 r.p.m.; (*b*) 25 r.p.m.; (*c*) 15 r.p.m.
10. 0·0061 in. 11. 160 r.p.m. same direction. 12. $51\frac{3}{47}$.
13. 29·6 same direction. 14. B 70; C 200.
16. 12, 48, and 15, 45 teeth. 17. 4·95, say 5.
18. B 15; G $32\frac{8}{9}$; 3,100 lb.-in.
19. $104\frac{6}{11}$ revs. in same direction as A.
20. $98\frac{42}{71}$ in opposite direction. 21. C 21; D 63 teeth.
22. $11\frac{1}{9}$ revs. in same direction as A. 23. $31\frac{7}{8}$.
24. 60 r.p.m. in opposite direction.
25. 231 r.p.m.; 408 r.p.m.
27. C 13; D 52.
28. 131·6; 36·85 r.p.m.
29. 2·54 r.p.m. in opposite direction.
30. $2\frac{1}{2}$ r.p.m., 95 r.p.m.
31. 3·08.

VIII (pp. 214–217)

2. 1,211 lb.-in.; 136·4 lb.-in.; 44·1 per cent. 3. (*a*) 26·1; (*b*) 25·4.
4. 11·52. 5. 0·0952; 0·3808. 6. 31·5 per cent.; 20·7 lb.-ft.
7. 2·13. 8. 1·35 tons. 9. (*a*) 0·508; (*b*) 0·381.
10. 1·40. 11. (*a*) 20·6 in.; (*b*) 4. 12. 90·9. 13. 31·1.
14. (*a*) 1·143 tons; (*b*) 0·538 ton. 15. 1·28. 16. 87·8.
18. 2·3°; 15,240 lb.-ft. 19. 58·5 lb.
20. 18·45 lb. 21. 60·2 lb.-ft.

IX (pp. 238–243)

1. (*a*) 12 in.; (*b*) 11·917 in.; (*c*) 11·43 in. 2. 26 ft. 0·6 in.
3. 6·59 H.P. 4. 61·3 lb. per sq. in.; 15·3 lb. per in. width.
5. (*a*) $l = 331{\cdot}3$ in.; $d = 29{\cdot}1$ in.; (*b*) $l = 334{\cdot}6$ in.; $d = 28$ in.
6. 12·56; 9·36. 7. 81 lb.; 28·4 lb.

8. 9 and 18 in.; 11·13 and 15·9 in.; 12·78 and 14·2 in.

9. 4·5, 5·6, and 6·48 in. dia. on countershaft; 9·9, 8·8, and 7·92 on lathe.

10. 3·6, say 4. 11. 3·16 in.

12. 12 and 20 in.; 4⅜ in. assuming $\frac{T_1}{T_2} = 2$.

13. 10·56, 12·5, and 13·88 in. dia. on driving shaft; 11·32, 9·38, and 8 in. on driven shaft; l = 323·2 in. assuming smallest pulley 8 in. dia., and $\frac{T_1}{T_2} = 2$.

14. 107·5 lb.-ft.; 59·3 lb.-ft. 15. 2·07. 16. 240·3 lb.
18. 2·94 in. 19. 47 ft. 8·35 in. 20. 12·0.
21. 5·12 in.; 168 lb.; 224 lb. 22. 2·57.
23. 21·76 and 32·64 in. 24. 16·4 in.
25. 12·2 in.; 37·9. 26. 282 lb.; 28·6.
27. 10·84; 16·9 per cent. 28. 3·08 in.
29. 576 lb., 41·8 per cent.

X (pp. 258–262)

1. 3,600 ft.-tons; 60·46 and 59·44 r.p.m.
2. 17,530 ft.-lb. 3. 820 lb.-ft. 4. 970 lb.
5. 261,000 ft.-lb.; 237 H.P.
6. Out, 16,210 and 21,476 lb.; In, 17,524 and 22,790 lb.
7. 35·1; 24·8 tons. 8. 1,060 lb.-ft. 9. 121·07; 118·93 r.p.m.
10. (*a*) 20,145 lb.; (*b*) 12,120 lb.-ft.; (*c*) 9·67 rads. per sec. per sec.
11. 363; 891 ft. per sec. per sec.; 969 lb.; 34·3 lb. per sq. in.
12. (*a*) 80·5 lb.-ft. units; (*b*) 39,700 ft.-lb.
13. 0·854 rad. per sec. per sec. 14. 5,370 lb. 15. 108.
16. 6,000 lb. 17. 308 lb.-ft. units. 19. 1·12 tons.
20. 26·7 lb. per sq. in.; 16 lb. per sq. in.
21. $E = 777W$; 17·55 tons-ft.
22. 12,200 lb.; 7,420 lb.-ft.; 353 H.P.
23. 36·7 sq. in. 24. 9,950 lb. 25. 33·9 and 21·95 lb. per sq. in.
26. 7·9 r.p.m. 27. 969 lb. 28. 7,180 lb.

XI (pp. 294–298)

1. 57·9 lb.; (*a*) 216 (*b*) 183·5 r.p.m. 2. 25 lb.; 123·3 r.p.m.
3. 15·2 lb.; 130·8 r.p.m. 6. 92·4; 76·4 r.p.m.
7. 310; 353 r.p.m. 8. 306·7; 290·9 r.p.m.
9. (*a*) 289·5 and 283·5; (*b*) 300 and 300·5; (*c*) 306·5 and 310 r.p.m.
10. 101 lb. per in.; 3·85 in.
11. 79·8 lb. per in.; (*a*) 306 (*b*) 294 r.p.m. 12. 254·5 r.p.m.
13. 264 and 227 r.p.m.; 249 and 237·5 r.p.m.
14. 119·5; 129·3 r.p.m.
15. (*a*) 34·5 and 31·5 r.p.m.; (*b*) 45·6 and 45 r.p.m.
16. 47·7 lb.; 215 and 184 r.p.m. 17. 207·5; 192·1 r.p.m.
18. 1·01 lb. 19. 1·087 in.; 180·8–170·8 r.p.m.

20. 6·36 ; 3·6 per cent. 21. 39·1 lb. per in.; 63·6 lb. per in.
22. 179 r.p.m.; 16·5 r.p.m.; 13·2 lb.
23. 8·81 ; 5 26 r.p.m. 24. 0·0739 ; 204 r.p.m.
25. 43·2 lb. per in.; 2·08 in. 8·6 r.p.m.

XII (pp. 316–320)

1. 1,726 lb.; 108° 26′ to 20 lb. mass; 50·6 lb.
2. 10·5 lb.; 147°, 224½°, and 18° with *A*.
4. 90 lb.; 843 lb. 5. 1·65 lb.; 118·5° with *A*.
7. 310 lb. at 157½° with adjacent crank. 8. 151·5° ; 0 692 ton.
9. 12·24 and 2·76 tons. 10. 3·21 ; 2·07 tons.
11. (*a*) *A–B* 188·5° ; *A–D* 317·3° (*b*) 7·65 in.; (*c*) 19·25 lb.
13. 39,000 lb.-ft.; *A*, 160 lb. at 43° with 3 ; *B*, 160 lb. at 43° with 1.
14. 3,800 lb. 15. 7·6 lb.; 4·52 lb. 16. 37·0 in.; 4,020 lb.-in.
17. 385 lb.; 184° 46′ with adjacent crank. 18. 1,000 lb.
19. (*a*) 3,710 lb.; (*b*) 3,450 lb.; (*c*) 3,315 lb.-ft. 20. 1·73 tons.
21. *A* 1½ ft. from *B* ; *D* 0·89 ft. from *C* ; *A* at an angle of 106° with *C* and 29° with *D*.
22. *A–B*, 124° ; *B–C*, 141½° ; *C–A*, 94½° ; 91·8 lb.-ft.; 5·74 lb.
23. 2 31 ft. from *C* ; 24·7 lb.; 79° with *C*.

XIII (pp. 346–351)

3. 3·43 ft. per sec. 9. 1·16 in.; 30·5°. 10. 86°.

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